8 Antennas

In Chapter 6 we found that electromagnetic waves can propagate in free space, and in Chapter 7 we saw that they can be guided in circuits. Now we need to make the important connection between these descriptions: how do electrons in a circuit excite waves in free space, and *vice versa*? This is the essential job of an *antenna*.

There are almost as many antenna designs as there are antenna designers. In theory, we've covered everything that's needed to understand them. In practice, as the size of a circuit grows to become comparable to the wavelenth of the signal that it carries, the explanation of its function can appear to grow from trivial to impenetrable because of the many tricks and approximations that are needed in this difficult regime. As we progress, it will be helpful to keep track of the purpose of an antenna: to match the impedance of a circuit with that of free space, for radiation with the desired frequency, orientation, and polarization. If this is done properly the antenna will efficiently couple the signal across rooms or solar systems; if it is not, then the antenna will serve as a reflector to heat up the signal generator.

8.1 TIME-DEPENDENT POTENTIALS

The electromagnetic potentials will be convenient for studying antennas, but we'll need to understand their behavior for fields that vary in time. Plugging the definition of the vector potential $\vec{B} = \nabla \times \vec{A}$ into $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$ gives

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \quad . \tag{8.1}$$

Equation (6.13) showed that this will hold if the argument is the gradient of a function

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi \quad , \tag{8.2}$$

or

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \quad . \tag{8.3}$$

For a time-dependent field, we have to add the derivative of the vector potential to the gradient of the scalar potential to find $\vec{E}$.

Since $\vec{B}$ is the curl of $\vec{A}$, we can add the gradient of another potential to $\vec{A}$ without

changing the magnetic field:

$$\vec{A} \rightarrow \vec{A} + \nabla\psi \quad . \tag{8.4}$$

This will change $\vec{E}$, however, but that can be canceled if Φ is replaced with

$$\Phi \rightarrow \Phi - \frac{\partial\psi}{\partial t} \quad . \tag{8.5}$$

This is called a *gauge transformation* and can be used to simplify the vector and scalar potentials.

In terms of the potentials, Poisson's equation $\nabla \cdot \vec{D} = \rho$ becomes in free space

$$\nabla \cdot \epsilon_0 \left(-\nabla\Phi - \frac{\partial\vec{A}}{\partial t} \right) = \rho$$

$$\nabla^2\Phi + \frac{\partial}{\partial t}\nabla \cdot \vec{A} = -\frac{1}{\epsilon_0}\rho \quad . \tag{8.6}$$

A second relationship between $\vec{A}$ and Φ comes from the Maxwell equation for the curl of $\vec{H}$, which in free space is

$$\frac{1}{\mu_0}\nabla \times \vec{B} = \vec{J} + \epsilon_0\frac{\partial E}{\partial t}$$

$$\underbrace{\nabla \times (\nabla \times \vec{A})}_{\nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}} = \mu_0\vec{J} + \underbrace{\mu_0\epsilon_0}_{c^{-2}}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial\vec{A}}{\partial t} \right)$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} \right) = -\mu_0\vec{J} \quad . \tag{8.7}$$

If we transform to the *Lorentz gauge* in which the potentials are related by

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0 \quad , \tag{8.8}$$

equations (8.6) and (8.7) take on the satisfying form of wave equations excited by the charge and current respectively,

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{1}{\epsilon_0}\rho$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \quad . \tag{8.9}$$

These are linear partial differential equations. If we take a periodic time dependence $\Phi(\vec{x}, t) = \Phi(\vec{x})e^{i\omega t}$, with likewise for $\vec{A}$, ρ, and $\vec{J}$, the time dependence cancels out:

$$\nabla^2\Phi(\vec{x}) + k^2\Phi(\vec{x}) = -\frac{1}{\epsilon_0}\rho(\vec{x})$$

$$\nabla^2\vec{A}(\vec{x}) + k^2\vec{A}(\vec{x}) = -\mu_0\vec{J}(\vec{x}) \quad , \tag{8.10}$$

where $k^2 = \omega^2/c^2$. The solution for arbitrary time dependence can be found by Fourier superposition.

In Chapter 6 we solved Laplace's equation with a Green's function; we can use the same technique to find a Green's function $G(\vec{x}, \vec{x}')$ that solves *Helmholtz' equation*

$$\nabla^2 G(\vec{x}, \vec{x}') + k^2 G(\vec{x}, \vec{x}') = -4\pi\delta(|\vec{x} - \vec{x}'|) \quad . \tag{8.11}$$

Define $r = |\vec{x} - \vec{x}'|$. Since the delta function depends only on r, by symmetry G will not depend on the angle between $\vec{x}$ and $\vec{x}'$, leaving only the radial term in the expansion of the Laplacian in spherical coordinates (equation 6.20)

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) + k^2 G = -4\pi\delta(r) \quad . \tag{8.12}$$

For $r \neq 0$ this reduces to the homogeneous equation

$$\frac{\partial^2}{\partial r^2}(rG) + k^2 rG = 0 \quad , \tag{8.13}$$

which can immediately be solved to find

$$rG = \alpha e^{ikr} + \beta e^{-ikr} \tag{8.14}$$

or

$$G = \alpha\frac{e^{ikr}}{r} + \beta\frac{e^{-ikr}}{r} \tag{8.15}$$

for arbitrary constants α and β. In the limit $r \to 0$ the Green's function reduces to

$$G = \frac{\alpha + \beta}{r} \quad . \tag{8.16}$$

Since we know that

$$\nabla^2\frac{1}{r} = -4\pi\delta(r) \tag{8.17}$$

from equation (6.44), this means that the coefficients must satisfy $\alpha + \beta = 1$. The potentials are then found by integrating the Green's function over the source distributions

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x})\left(\alpha e^{ik|\vec{r}-\vec{x}|} + \beta e^{-ik|\vec{r}-\vec{x}|}\right)}{|\vec{r} - \vec{x}|}\,d\vec{x}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x})\left(\alpha e^{ik|\vec{r}-\vec{x}|} + \beta e^{-ik|\vec{r}-\vec{x}|}\right)}{|\vec{r} - \vec{x}|}\,d\vec{x} \quad . \tag{8.18}$$

The exponentials in the numerators represent a phase shift in the propagation of a spherical wave from the sources. Equating these with the time delay $e^{\pm ik\Delta x} = e^{i\omega\Delta t}$ shows that $\Delta t = \pm k\Delta x/\omega = \pm\Delta x/c$. There's a problem here: the positive solution corresponds to a wave that travels backwards rather than forwards in time. Since as far as we know that's not possible, we'll drop the *advanced potential* solution and stick with the causal *retarded potential* solution with $\beta = 1$ [Anderson, 1992]:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x})\,e^{-ik|\vec{r}-\vec{x}|}}{|\vec{r} - \vec{x}|}\,d\vec{x}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x})\,e^{-ik|\vec{r}-\vec{x}|}}{|\vec{r} - \vec{x}|}\,d\vec{x} \quad . \tag{8.19}$$

In homogeneous media these are modified by including the relative permittivity and permeability.

Substituting the periodic time dependence of $\vec{A}$ and Φ into the Lorentz gauge relates them by

$$\begin{aligned} \nabla \cdot \vec{A} &= -\mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} \\ &= -i\omega \mu_0 \epsilon_0 \Phi \quad . \end{aligned} \tag{8.20}$$

This in turn means that a periodic electric field can be written in terms of the vector potential alone,

$$\begin{aligned} \vec{E} &= -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \\ &= -\nabla \Phi - i\omega \vec{A} \\ &= \frac{1}{i\omega \mu_0 \epsilon_0} \nabla(\nabla \cdot \vec{A}) - i\omega \vec{A} \quad . \end{aligned} \tag{8.21}$$

Therefore if we can solve equation (8.19) for $\vec{A}$ then everything else can be found from it.

There's one final subtlety in using time-dependent electromagnetic potentials. We've written them with impunity as complex quantities $A = A_0 e^{i\theta_A} e^{i\omega t}$, with the phase angle keeping track of the *sin* and *cos* components independently. This trick works for linear transformations like addition and integration, but it fails for nonlinear operations like multiplication which will mix the real and imaginary parts. Consider the time-average of the real parts of two complex quantities A and B, which can be found from the sums of their complex conjugates:

$$\begin{aligned} \langle \text{Re}[A]\text{Re}[B] \rangle &= \left\langle \frac{(A + A^*)}{2} \frac{(B + B^*)}{2} \right\rangle \\ &= \frac{1}{4} \langle AB + A^*B + AB^* + A^*B^* \rangle \\ &= \frac{1}{4} A_0 B_0 \left\langle e^{i(\theta_A + \theta_B + 2\omega t)} + e^{-i(\theta_A + \theta_B + 2\omega t)} + e^{i(\theta_A - \theta_B)} + e^{-i(\theta_A - \theta_B)} \right\rangle \\ &= \frac{1}{2} A_0 B_0 [\cos(\theta_A - \theta_B) + \underbrace{\langle \cos(\theta_A + \theta_B + 2\omega t) \rangle}_{0}] \\ &= \frac{1}{2} A_0 B_0 \cos(\theta_A - \theta_B) \quad . \end{aligned} \tag{8.22}$$

This does not equal $\text{Re}\langle AB \rangle$, but notice that it is the same as

$$\begin{aligned} \frac{1}{2}\text{Re}[A^*B] &= \frac{1}{4}(A^*B + AB^*) \\ &= \frac{1}{4} A_0 B_0 \left(e^{i(-\theta_A + \theta_B)} + e^{i(\theta_A - \theta_B)} \right) \\ &= \frac{1}{2} A_0 B_0 \cos(\theta_A - \theta_B) \\ &= \langle \text{Re}[A]\text{Re}[B] \rangle \end{aligned}$$

$$= \frac{1}{2} A_0 B_0 \cos(\theta_B - \theta_A)$$

$$= \frac{1}{2} \text{Re}[AB^*] \quad . \tag{8.23}$$

This means that we can continue our practice of finding observable values by taking the real part at the end of a calculation involving complex quantities if the time average of a product is replaced with half of one factor times the complex conjugate of the other. The most important place where this will be needed is in evaluating the Poynting vector

$$\langle \vec{P} \rangle = \langle \vec{E} \times \vec{H} \rangle$$

$$= \frac{1}{2} \text{Re} \left[\vec{E} \times \vec{H}^* \right] \tag{8.24}$$

for periodic fields.

8.2 DIPOLE RADIATION

8.2.1 Infinitesimal Length

We're now equipped to find the fields radiated by oscillating charges and currents. The simplest case is a thin wire carrying a periodic current with a constant amplitude $\vec{J} = I_0 \delta(x, y) \hat{z}$. If we further assume that the wire has an infinitesimal length d in the $\hat{z}$ direction, the vector potential can be read off from equation (8.19) to be

$$\vec{A}(r) = \mu_0 \frac{I_0 d e^{-ikr}}{4\pi r} \hat{z} \quad , \tag{8.25}$$

with $r = |\vec{x} - \vec{x}'|$ the distance from the source. This is called a *Hertz dipole* because the current must be associated with periodically-varying point charges at the ends of the wire. Since in spherical coordinates (r, θ, φ) the unit normal $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$, the vector potential around the wire is

$$A_r = \mu_0 \frac{I_0 d e^{-ikr}}{4\pi r} \cos\theta \quad A_\theta = -\mu_0 \frac{I_0 d e^{-ikr}}{4\pi r} \sin\theta \quad . \tag{8.26}$$

The only non-zero term in equation (6.21) is

$$\vec{B} = \nabla \times \vec{A}$$

$$= \frac{1}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial \theta} A_r \right] \hat{\varphi} \quad , \tag{8.27}$$

giving

$$B_\varphi = \frac{\mu_0 I_0 d}{4\pi} e^{-ikr} \left(\frac{ik}{r} + \frac{1}{r^2} \right) \sin\theta \quad . \tag{8.28}$$

Equation (8.21) can be used to find the corresponding electric field (Problem 8.1),

$$E_\theta = \frac{I_0 d}{4\pi} e^{-ikr} \left(\frac{i\omega\mu_0}{r} + \frac{1}{r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{1}{i\omega\epsilon_0 r^3} \right) \sin\theta$$

$$E_r = \frac{I_0 d}{4\pi} e^{-ikr} \left(\frac{2}{r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{2}{i\omega\epsilon_0 r^3} \right) \cos\theta \quad . \tag{8.29}$$

There are three exponents for the radial dependence. The r^{-3} term will dominate as $r \to 0$, reducing to the electric field of a static dipole with $I_0 d = p$ (equation 6.70). This is the *near-field* or *static* zone. The r^{-1} term will dominate as $r \to \infty$, the *far-field* or *radiation* zone. And the r^{-2} is significant in the intermediate *induction* zone. Taking only the r^{-1} terms significant in the far field, the Poynting vector is

$$
\begin{aligned}
\langle \vec{P} \rangle &= \frac{1}{2} \mathrm{Re}[\vec{E} \times \vec{H}^*] \\
&= \hat{r} \frac{1}{2} \mathrm{Re}[E_\theta H_\varphi^*] \\
&= \hat{r} \frac{I_0^2 k^2 d^2}{32\pi^2 r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sin^2 \theta \quad .
\end{aligned}
\tag{8.30}
$$

This can be integrated over a sphere to find the total energy radiated,

$$
\begin{aligned}
W &= \int \vec{P} \cdot d\vec{A} \\
&= \int_0^{2\pi} \int_0^\pi P_r r^2 \sin\theta \, d\theta \, d\varphi \\
&= \int_0^\pi 2\pi P_r r^2 \sin\theta \, d\theta \\
&= \int_0^\pi 2\pi \frac{I_0^2 k^2 d^2}{32\pi^2 r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sin^2\theta r^2 \sin\theta \, d\theta \\
&= \frac{I_0^2 k^2 d^2}{16\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \int_0^\pi \sin^3\theta \, d\theta \\
&= \frac{I_0^2 k^2 d^2}{12\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \\
&= \frac{I_0^2 \pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{d}{\lambda} \right)^2 \quad .
\end{aligned}
\tag{8.31}
$$

The r^{-1} dependence of the leading terms in $\vec{E}$ and $\vec{H}$ give an r^{-2} decay that is canceled by the r^2 surface area, leaving the total radiated power over all directions independent of the distance.

In a periodically driven resistor, the power consumption is

$$
\begin{aligned}
W &= \langle I^2 R \rangle \\
&= I_0^2 \underbrace{\langle \sin^2(\omega t) \rangle}_{1/2} R \\
&= \frac{I_0^2 R}{2} \quad .
\end{aligned}
\tag{8.32}
$$

If we turn this around to define a resistance, the power radiated from the dipole is related to the magnitude of the current that excites it by

$$
\begin{aligned}
R_{\text{rad}} &= \frac{2W}{I_0^2} \\
&= \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{d}{\lambda} \right)^2 \quad .
\end{aligned}
\tag{8.33}
$$

This is called the *radiation resistance*. If it is small, the real impedances in the circuit feeding the antenna will dominate, leaving most of the energy dissipated there rather than radiated as desired. That will certainly be true here because d is infinitesimal.

8.2.2   Finite Length

A more effective antenna drives a resonant mode of a longer dipole so that the oscillatory energy can stay in the antenna. Following a similar calculation to the waveguide modes we found in the last chapter, the current distribution for the lowest-order mode of a center-fed dipole of length d

$$\vec{J}(\vec{x}) = I_0 \, \delta(x, y) \sin \left[k \left(\frac{d}{2} - |z| \right) \right] \hat{z} \tag{8.34}$$

has a maximum at the center and vanishes at the ends [Jackson, 1999]. This neglects the small corrections to the current distribution due to radiation damping and the finite transverse thickness of the wire.
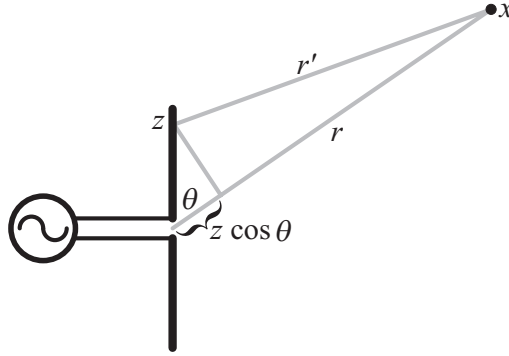

Figure 8.1. Center-fed dipole antenna.

The geometry is shown in Figure 8.1. We'll be interested in the far-field radiation pattern, for which $r \gg z$, therefore the path difference as a function of position along the antenna can be approximated by $r' \approx r - z \cos \theta$. This difference will be sigificant in the phase of the spherical wave $e^{ikr'}$ over the antenna, but not in the decay of the amplitude r'^{-1} which will be a small change in the large radius. With these approximations equation (8.19) can easily be integrated to find

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} I_0 \sin \left[k \left(\frac{d}{2} - |z| \right) \right] \frac{e^{ik(r - z \cos \theta)}}{r} \, dz$$

$$= \frac{\mu_0}{2\pi} I_0 \frac{e^{ikr}}{kr} \frac{\cos \left(\frac{kd}{2} \cos \theta \right) - \cos \left(\frac{kd}{2} \right)}{\sin^2 \theta} \hat{z} \quad . \tag{8.35}$$

Referring back to the curl in spherical coordinates, equation (6.21), all of the terms have an r^{-1} dependence and hence can be ignored except for two:

$$\vec{B} = \nabla \times \vec{A} \rightarrow -\frac{1}{r} \frac{\partial}{\partial r} (r A_\varphi) \, \hat{\theta} + \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \, \hat{\varphi} \quad . \tag{8.36}$$

Writing $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ and taking the curl,

$$\vec{B} = -\frac{\mu_0}{2\pi}I_0\frac{ie^{ikr}}{r}\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\,\hat{\varphi}\quad. \tag{8.37}$$

From this the electric field can be found,

$$\vec{E} = -\sqrt{\frac{\mu_0}{\epsilon_0}}\hat{k}\times\vec{H}$$

$$= -\frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}I_0\frac{ie^{ikr}}{r}\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\,\hat{\theta}\quad, \tag{8.38}$$

and the Poynting vector

$$\langle\vec{P}\rangle = \frac{1}{2}\text{Re}[\vec{E}\times\vec{H}^*]$$

$$= \frac{1}{8\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I_0^2}{r^2}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2\,\hat{r}\quad. \tag{8.39}$$

Contours of constant $\vec{P}$ are plotted in cross-sectional slice in Figure 8.2 for $kd/2 = \pi/2$, a *half-wave* dipole antenna. The difference between these curves and the circles that would describe spherical waves represents *directivity*: the antenna sends more energy in a desired direction at the expense of less in others.
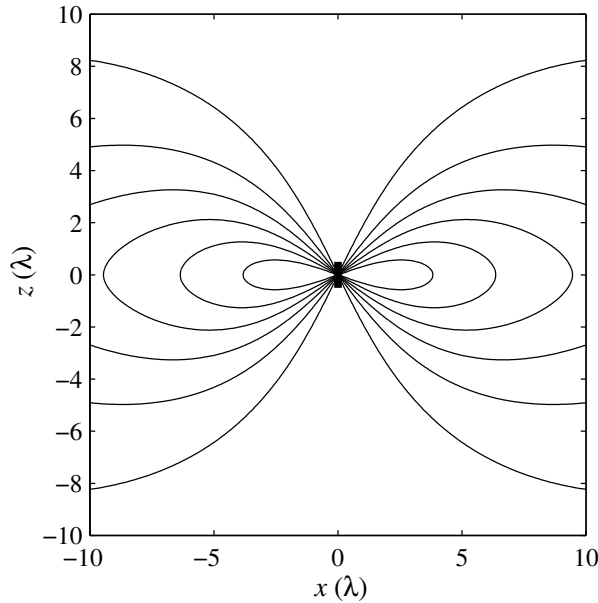

Figure 8.2. Radiation from a half-wave dipole.

The total power emitted is

$$W = \int\vec{P}\cdot d\vec{A}$$

$$= \int_0^{2\pi} \int_0^{\pi} P_r r^2 \sin\theta \ d\theta \ d\varphi$$

$$= \int_0^{\pi} 2\pi P_r r^2 \sin\theta \ d\theta$$

$$= \frac{I_0^2}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \int_0^{\pi} \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \ d\theta \quad . \tag{8.40}$$

The integral can be written in terms of special functions or evaluated numerically; for $kd = \pi$ it is

$$W \approx \frac{I_0^2}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \ 1.22 \quad . \tag{8.41}$$

This gives a radiation resistance of

$$R_{\mathrm{rad}} = \frac{2W}{I_0^2}$$

$$= \frac{2.44}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\approx 73 \ \Omega \quad , \tag{8.42}$$

quite a difference from the infinitesimal case. This value explains the prevalence of 75 Ω coaxial cables, although the radiation resistance will vary from this value because of loading from nearby objects (such as the Earth), and have an imaginary component due to the antenna's non-infinitesimal dimensions.

8.3 DUALITY AND RECIPROCITY

Maxwell's equations in free space, equation (6.97), have an interesting symmetry: they are left unchanged if $\vec{E}$ is replaced with $\vec{H}$, $\vec{H}$ replaced with $-\vec{E}$, and μ and ϵ are interchanged. This *duality* relationship means that any free-space solution immediately provides a second dual one with the electric and magnetic fields swapped. Since we've already seen that the static field of an electrical dipole (equation 6.70) has the same form as the static field of a magnetic dipole (equation 6.79), this means that the roles of the electric and magnetic fields can be exchanged in equation (8.29) to find the radiation from a magnetic dipole source.

A deeper symmetry starts with the rather formal observation that any two sets of fields $(\vec{E}_1, \vec{H}_1)$ and $(\vec{E}_2, \vec{H}_2)$ that are periodic with a frequency ω must satisfy

$$\nabla \cdot \left(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1\right)$$

$$= \vec{H}_2 \cdot \left(\nabla \times \vec{E}_1\right) - \vec{E}_1 \cdot \left(\nabla \times \vec{H}_2\right) - \vec{H}_1 \cdot \left(\nabla \times \vec{E}_2\right) + \vec{E}_2 \cdot \left(\nabla \times \vec{H}_1\right)$$

$$= \vec{H}_2 \cdot \left(-\frac{\partial \vec{B}_1}{\partial t}\right) - \vec{E}_1 \cdot \left(\vec{J}_2 + \frac{\partial \vec{D}_2}{\partial t}\right) - \vec{H}_1 \cdot \left(-\frac{\partial \vec{B}_2}{\partial t}\right) + \vec{E}_2 \cdot \left(\vec{J}_1 + \frac{\partial \vec{D}_1}{\partial t}\right)$$

$$= -i\omega \vec{H}_2 \cdot \vec{B}_1 - \vec{E}_1 \cdot \vec{J}_2 - i\omega \vec{E}_1 \cdot \vec{D}_2 + i\omega \vec{H}_1 \cdot \vec{B}_2 + \vec{E}_2 \cdot \vec{J}_1 + i\omega \vec{E}_2 \cdot \vec{D}_1$$

$$= -i\omega\mu \vec{H}_2 \cdot \vec{H}_1 - \vec{E}_1 \cdot \vec{J}_2 - i\omega\epsilon \vec{E}_1 \cdot \vec{E}_2 + i\omega\mu \vec{H}_1 \cdot \vec{H}_2 + \vec{E}_2 \cdot \vec{J}_1 + i\omega\epsilon \vec{E}_2 \cdot \vec{E}_1$$

$$= \vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2 \qquad (8.43)$$

for sources $\vec{J}_1$ and $\vec{J}_2$. If both sides are integrated over the volume of a sphere,

$$\int \left(\vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2 \right) dV = \int \nabla \cdot \left(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1 \right) dV \quad , \qquad (8.44)$$

Gauss' Law turns the right hand side into a surface integral,

$$\int \left(\vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2 \right) dV = \int \left(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1 \right) \cdot d\vec{A} \quad , \qquad (8.45)$$

and in the limit that the radius goes to infinity the near-field solutions will disappear, leaving $\vec{E}$ and $\vec{H}$ as the transverse components of an outgoing spherical wave with the local wave vector $\vec{k} = k\hat{r}$ pointing radially so that

$$\int \left(\vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2 \right) dV = \int \left[\vec{E}_1 \times \left(\sqrt{\frac{\epsilon_0}{\mu_0}} \vec{k} \times \vec{E}_2 \right) - \vec{E}_2 \times \left(\sqrt{\frac{\epsilon_0}{\mu_0}} \vec{k} \times \vec{E}_1 \right) \right] \cdot d\vec{A}$$
$$= 0 \quad . \qquad (8.46)$$

This is one form of the *Lorentz Reciprocity Theorem*, which provides a connection between the transmitting and receiving properties of an antenna.

The effective circuit of an antenna used as a receiver is shown in Figure 8.3. The radiation induces a voltage V across the antenna terminals, which appears to the load as in ideal generator in series with the antenna's radiation resistance. Problem 8.3 will show that a maximum power of $W = |V|^2/8R_{\text{load}}$ is delivered to the load if $R_{\text{load}} = R_{\text{rad}}$.
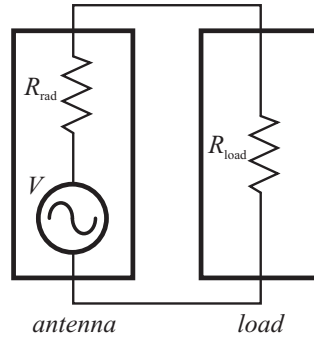

antenna   *load*

Figure 8.3. Effective circuit for a receiving antenna.

To find how V relates to the source current in the transmitting antenna, equation (8.46) can be applied to a pair of antennas. The integral will vanish except where the currents are non-zero, which can easily be evaluated for an infinitesimal dipole,

$$\int \vec{E}_2 \cdot \vec{J}_1 \ dV = \int \vec{E}_1 \cdot \vec{J}_2 \ dV$$
$$\int \vec{E}_2(\vec{x}_1) \cdot d\vec{l} \ J_1 \ dA = \int \vec{E}_1(\vec{x}_2) \cdot d\vec{l} \ J_2 \ dA$$
$$V_2(\vec{x}_1) I_1 = V_1(\vec{x}_2) I_2$$
$$\frac{I_1}{V_1(\vec{x}_2)} = \frac{I_2}{V_2(\vec{x}_1)} \quad . \qquad (8.47)$$

The ratio of the source current I_1 in antenna 1 to the voltage $V_1(\vec{x}_2)$ it induces in antenna 2 is equal to the ratio with the radiation going in the opposite direction. This result, called *reciprocity*, can be extended to apply to arbitrary pairs of antennas. This is surprising: antennas usually receive far-field plane-wave radiation, but they radiate solutions with much more complex near-field patterns; there's no *a priori* reason to expect any symmetry between these processes.

Reciprocity provides useful connections among many antenna properties. The *gain G* of an antenna is defined to be the maximum value of the Poynting vector, evaluated on the surface of a unit sphere, divided by the total power radiated over the area of that sphere

$$G \equiv \max_{\theta, \varphi} \frac{P(r = 1, \theta, \varphi)}{W/4\pi} \quad . \tag{8.48}$$

Directional antennas can have gains much larger than 1, the value for an isotropic radiator.

If an antenna transmits a power W_1, the received far-field power at a second antenna W_2 will be

$$W_2 = A_2 \frac{G_1}{4\pi r^2} W_1 \quad . \tag{8.49}$$

The transmitted power decreases as a spherical wave but is increased by the gain, and A defines the receiving antenna's *area* which is the effective cross-section it presents to capture the incoming power density. The ratio of the transmitted and received power is thus

$$\frac{W_2}{W_1} = \frac{1}{4\pi r^2} A_2 G_1 \quad . \tag{8.50}$$

If antenna 1 receives and antenna 2 transmits the roles are reversed:

$$\frac{W_1}{W_2} = \frac{1}{4\pi r^2} A_1 G_2 \quad , \tag{8.51}$$

but because of reciprocity these ratios must be equal:

$$\begin{aligned} \frac{W_1}{W_2} &= \frac{W_2}{W_1} \\ A_1 G_2 &= A_2 G_1 \\ \frac{A_1}{G_1} &= \frac{A_2}{G_2} \quad . \end{aligned} \tag{8.52}$$

Since these antennas are arbitrary, we conclude that the ratio of an antenna's area to its gain is a constant independent of its design. In Problem 8.4 we'll see that this value is $\lambda^2/4\pi$.

These concepts of gain and area are essential to the design of useful RF links: the overall SNR can be improved by increasing the transmitter power, the transmitting antenna's gain, the receiving antenna's area, or the receiving amplifier's sensitivity. These present very different costs in the system's size/weight/dissipation/complexity/expense that must be allocated to where they are best borne.

8.4 ANTENNA TYPES

Antenna gain serves to direct energy where it is wanted, saving power and reducing interference. But how is it increased? One approach is shown in Figure 8.4, which plots the in-plane far-field radiation pattern for arrays of dipoles, viewed from above. A single dipole radiates a spherical wave, with field components that depend on distance as e^{ikr}/r. If there is more than one dipole, and the spacing between them is small compared to the distance r, then as in Figure 8.1 the dependence of the amplitude on the relative distances can be ignored and the phase dependence approximated by summing over their locations y_n as $\sum_n e^{ik(r-y_n\cos\theta)}/r$. Normal to the axis of the array the dipoles add in phase, but if they are placed a half-wavelength apart they will interfere destructively in the transverse direction, resulting in increasingly directional radiation patterns as antennas are added. This is called a *broadside* array, because the beam comes from the broad side. Alternatively, if the relative phases of the antennas match the phase shift over their separation, then they will add constructively along their axis and destructively in the normal direction, giving an *end-fire* array. AM radio stations usually use one of these configurations, so the antenna orientation points to the market that they are aimed at. By actively varying these phases electronically rather than passively determining them by geometry, beams can be formed for pointing in *phased array* radars [Latha *et al.*, 2021], and for channel-sharing in *MIMO* (Multiple–Input Multiple-Output) [Lu *et al.*, 2014] antennas.
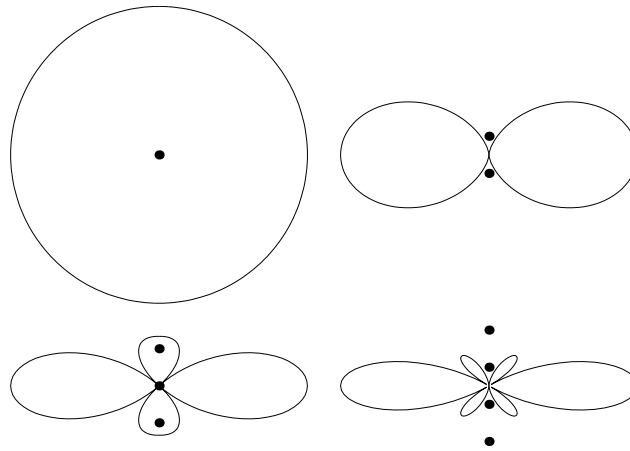

Figure 8.4. Looking down on antennas in a broadside array, showing the location of the antennas and contours of the field strength.

The *Yagi–Uda* array shown in Figure 8.5 cleverly obtains directionality with just one driven antenna, by parasitically exciting the rest of the array. Depending on the spacing, these secondary antennas can be *directors* that add constructively in a desired direction, or *reflectors* that add destructively in an undesired one.

Beyond dipoles, the enormous range of antenna designs reflects their many applications. A *micropatch* antenna uses a stripline to drive a resonant mode of a conductor over
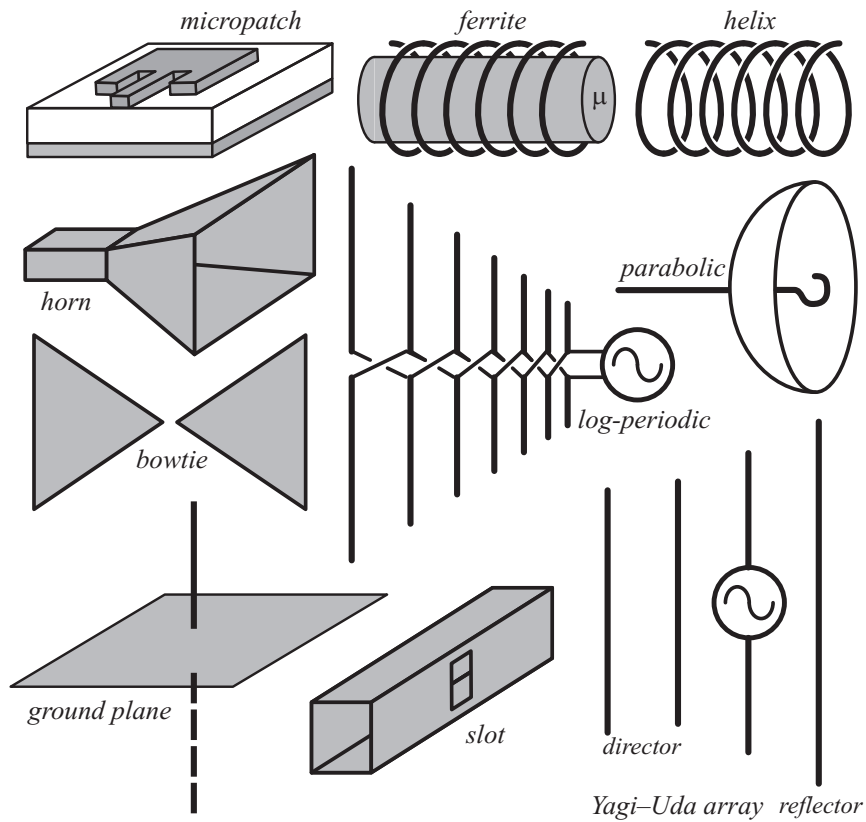
Figure 8.5. Some antenna types.

a ground plane. Because they have a planar structure, these are used where antennas must be conformal to a surface. The slots in the micropatch in Figure 8.5 provide impedance matching between the patch and the driving line. A *horn* matches the impedance between a waveguide and free space by increasing the area so gradually that there is no appreciable scattering. In the limit that the wavelength is small compared to the size of the antenna, the waves can be directed geometrically, as is done in the *parabolic* antennas that provide the large areas needed for weak satellite signals. Another way to increase the effective size of an antenna that's useful at low frequencies is to pull in more of the magnetic flux with a *ferrite* that provides permeability without conductivity (Chapter 13). These are found in AM receivers. At low frequencies the coil serves just to collect magnetic flux, but at much higher frequencies a traveling wave will follow it around. This is used in a *helical* antenna to launch and detect circularly-polarized radiation, eliminating the signal dependence on the relative orientation between the transmitter and the receiver, and providing two independent polarization channels that can be used in a single band.

Antennas are frequently called upon to operate over a range of frequencies. To be independent of wavelength, their shape must be invariant if it is rescaled. One kind of broadband antenna is the *bow-tie*, which unlike a dipole has a continuous spectrum of modes that extend further out as the frequency is decreased. A variant is the *log-peridioc* antenna, familiar on roofs as TV aerials, which is a finite segment from an infinite series

of dipoles with a geometrical scaling of their size and spacing, connected in an alternating sequence so that the gain is directed towards the tapered end. And then there are *fractal* antennas that have a self-similar scaling structure [Werner & Ganguly, 2003].

Antennas can also be defined by the imposition of boundary conditions. If a monopole antenna is placed above a ground plane, the induced image charge will match the radiation from a dipole. This can be used to synthesize extra antenna elements, and to shield an antenna from interfering materials (like the Earth). And if a hole is cut into a waveguide a wave will leak out, which can be understood as the radiation from the currents matching the boundary conditions. These are used in distributed *slot* antennas, for example to distribute RF signals in tunnels.

Antennas are often limited by applications to be much shorter than the wavelength. In that case they are electromagnetically *short*; impedance matching to free space is done with a tuned resonant *matching network* circuit.

Perhaps the most interesting antennas of all are the ones that make no (apparent) sense. These give an evolutionary search algorithm free reign to optimize a structure to meet desired specifications, using numerical simulations and experimental tests as error metrics to guide the search. The resulting shapes can have no recognizable logic, but nevertheless beat the performance of conventional antenna designs [Hornby *et al.*, 2011].

Finally, at the limit of antennas is a regime that has become accessible only relatively recently. The range from 0.1 to 10 THz is above what's been traditionally considered to be RF, but below what's been traditionally considered to be optical. Referred to as a *terahertz*, *submillimeter*, or *quasi-optical* regime, it shares aspects of both. It can be generated and detected either by mixing up lower frequency radio signal or by mixing down higher frequency optical signals, and it can be guided by both striplines and lenses [Mittleman, 2017]. It has important emerging applications in high-speed communications [Federici & Moeller, 2010], spectroscopy [El Haddad *et al.*, 2013], and astronomy [Kulesa, 2011].

8.5 SELECTED REFERENCES

[Ramo *et al.*, 1994] Ramo, Simon, Whinnery, John R., & Duzer, Theodore Van. (1994). *Fields and Waves in Communication Electronics*. 3rd edn. New York: Wiley.

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Everything you ever wanted to know about antennas, and more.

8.6 PROBLEMS

(8.1) Find the electric field for an infinitesimal dipole radiator.

(8.2) What is the magnitude of the Poynting vector at a distance of 1 km from an antenna radiating 1 kW of power, assuming that it is an isotropic radiator with a wavelength much less than 1 km? What is the peak electric field strength at that distance?

(8.3) For what value of R_{load} is the maximum power delivered to the load in Figure 8.3?

(8.4) For an infinitesimal dipole antenna, what are the gain and the area, and what is their ratio?