# 17 Constrained Optimization

best frequently has constraints

nutrition

groceries $\vec{g} \geq 0$

prices $\vec{p}$

price $\min_{\vec{g}} \vec{g} \cdot \vec{p}$

minimum requirements $\vec{m}$

nutrition value $\mathbf{N}$

$\mathbf{N} \cdot \vec{g} \geq \vec{m}$

defines linear program, LP

price may be a function of quantity, not linear

quadratic objective, quadratic program, QP

general case mathematical program

portfolios, routing airplanes, running a factory

program as plan, not computer program, can be same

electrical networks [Dennis, 1958]

routing [Kelly, 1991, Papadimitriou & Steiglitz, 1998]

flow control [Low *et al.*, 2002]

layering [Chiang *et al.*, 2007]

sorting

variables $\vec{x}$, objective minimize $f(\vec{x})$, constraints $\vec{c}(\vec{x})$

max = -min

slack variables to convert inequality to equality

$$c(\vec{x}) \geq 0 \tag{17.1}$$

replace with

$$\begin{aligned} c(\vec{x}) - s &= 0 \\ s &\geq 0 \end{aligned} \tag{17.2}$$

combinatorial $x$ equals 1 or -1 relaxed as algebraic constraint $(x^2 - 1)^2 = 0$

L1 norm

$$|\vec{x}|_1 = \sum_i |x_i| \tag{17.3}$$

compressed sensing, sparsity
non-differentiable
[Schmidt *et al.*, 2007]
$(x)_+ = \max(x, 0)$
$(x)_- = \max(-x, 0)$

$$|x| = (x)_- + (x)_+ \tag{17.4}$$

$$\begin{aligned} |x| &\approx |x|_\alpha \\ &= \frac{1}{\alpha}\left[\log\left(1 + e^{-\alpha x}\right) + \log\left(1 + e^{\alpha x}\right)\right] \end{aligned} \tag{17.5}$$

$$\frac{d|x|_\alpha}{dx} = \frac{1}{1 + e^{-\alpha x}} - \frac{1}{1 + e^{\alpha x}} \tag{17.6}$$

$$\frac{d^2|x|_\alpha}{dx^2} = \frac{2\alpha e^{\alpha x}}{(1 + e^{\alpha x})^2} \tag{17.7}$$

minimize for increasing $\alpha$

## 17.1 LAGRANGE MULTIPLIERS

single equality constraint $c(\vec{x}) = 0$
step in direction $\vec{d}$ to minimize $f$ while satisfying the constraint

$$\begin{aligned} 0 &= c(\vec{x} + \vec{\delta}) \\ &\approx c(\vec{x}) + \nabla c \cdot \vec{\delta} \\ &= \nabla c \cdot \vec{\delta} \end{aligned} \tag{17.8}$$

step also minimizes $f$

$$\begin{aligned} 0 &> f(\vec{x} + \vec{\delta}) - f(\vec{x}) \\ &\approx f(\vec{x}) + \nabla f \cdot \vec{\delta} - f(\vec{x}) \\ &= \nabla f \cdot \vec{\delta} \end{aligned} \tag{17.9}$$

if $\nabla c(\vec{x})$ and $\nabla f(\vec{x})$ aligned not possible to find a direction, hence $\vec{x}$ is a local minimizer
define *Lagrangian*

$$\mathcal{L} = f(\vec{x}) - \lambda c(\vec{x}) \tag{17.10}$$

solve for

$$
\begin{aligned}
0 &= \nabla \mathcal{L} \\
&= \nabla f - \lambda \nabla c
\end{aligned} \tag{17.11}
$$

multiple constraints
linear combination

$$
\nabla f(\vec{x}) = \sum_i \lambda_i \nabla c_i(\vec{x}) \tag{17.12}
$$

$$
f(\vec{x}) = \sum_i \lambda_i c_i(\vec{x}) \tag{17.13}
$$

gives $\vec{x}(\vec{\lambda})$, substitute into constraints to find $\vec{\lambda}$
inequality constraint

$$
\begin{aligned}
0 &\leq c(\vec{x} + \vec{\delta}) \\
&\approx c(\vec{x}) + \nabla c \cdot \vec{\delta}
\end{aligned} \tag{17.14}
$$

if constraint not active ($c > 0$), can just do gradient descent $\vec{\delta} = -\alpha \nabla f$
for an active constraint $\nabla f \cdot \vec{\delta} < 0$ and $\nabla c \cdot \vec{\delta} \geq 0$
define half-planes
no intersection if point in same direction $\nabla f = \lambda \nabla c$
same condition, but now $\lambda \geq 0$

## 17.2 OPTIMALITY

first-order
equality constraints $c_i(\vec{x}), i \in \mathcal{E}$
inequality constraints $c_i(\vec{x}), i \in \mathcal{I}$
inactive constraint $\lambda_i = 0$
complementarity: $\lambda_i c_i = 0$: Lagrange multiplier only non-zero when constraint is active, otherwise reduces to gradient descent

$$
\begin{aligned}
\nabla_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) &= 0 \\
c_i(\vec{x}) &= 0 \quad (i \in \mathcal{E}) \\
c_i(\vec{x}) &\geq 0 \quad (i \in \mathcal{I}) \\
\lambda_i &\geq 0 \quad (i \in \mathcal{I}) \\
\lambda_i c_i(x) &= 0
\end{aligned} \tag{17.15}
$$

Karush-Kuhn-Tucker (KKT) conditions
necessary
second order: positive definite Lagrangian Hessian
sensitivity

replace $c(x) = 0$ with $c(x) = \epsilon$
minimizer $\vec{x}$ goes to $\vec{x}_\epsilon$

$$\begin{aligned} f(\vec{x}_\epsilon) - f(\vec{x}) &\approx \nabla f \cdot (\vec{x}_\epsilon - \vec{x}) \\ &= \lambda \nabla c \cdot (\vec{x}_\epsilon - \vec{x}) \\ &\approx \lambda \left(c(\vec{x}_\epsilon) - c(\vec{x})\right) \\ &= \lambda\epsilon \\ \frac{df}{d\epsilon} &= \lambda \end{aligned} \tag{17.16}$$

shadow prices: change in utility per change in constraint
$\vec{x}$ primal $\lambda$ dual
multi-objective
Pareto
not possible to improve one constraint without making others worse
defines Pareto frontier
can combine in multi-objective function with relative weights

## 17.3 SOLVERS

### 17.3.1 Penalty

penalty
combine

$$\mathcal{F} = f(\vec{x}) + \frac{\mu}{2} \sum_i c_i^2(\vec{x}) \tag{17.17}$$

$$\frac{\partial \mathcal{F}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \mu \sum_i c_i \frac{\partial c_i}{\partial x_j} \tag{17.18}$$

$$\mathcal{L} = f(\vec{x}) - \sum_i \lambda_i c_i(\vec{x}) \tag{17.19}$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_i \lambda_i \frac{\partial c_i}{\partial x_j} \tag{17.20}$$

effectively taking $c_i = -\lambda_i/\mu$
solving a different problem
driven to 0 as $\mu \to \infty$
large $\mu$ ill-conditioned
nonsmooth penalty

$$\mathcal{F} = f(\vec{x}) + \mu \sum_{i\in E} |c_i(\vec{x})| + \mu \sum_{i\in I} [c_i(\vec{x})]_- \tag{17.21}$$

exact for large enough $\mu$ [Nocedal & Wright, 2006]

non-differentiable
sub-gradient
approximate (17.5)
Newton steps, increase

### 17.3.2 Augmented Lagrangian

augmented Lagrangian

$$\mathcal{L} = f(\vec{x}) - \sum_i \lambda_i c_i(\vec{x}) + \frac{\mu}{2} \sum_i c_i^2(\vec{x}) \tag{17.22}$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_i \lambda_i \frac{\partial c_i}{\partial x_j} + \mu \sum_i c_i \frac{\partial c_i}{\partial x_j} \tag{17.23}$$

$\lambda_i^* = \lambda_i - \mu c_i$
$c_i = (\lambda_i - \lambda_i^*)/\mu$
vanishes much faster, as Lagrange multiplier estimates converge
$\lambda_i^{(n+1)} = \lambda_i^{(n)} - \mu c_i$
minimize $\vec{x}$, update $\lambda$, increase $\mu$

### 17.3.3 Interior Point

inequality constraints
can combine for bound, equality constraints
interior point
directly solve KKT system of equations
avoid boundaries
primal-dual

$$\begin{aligned} \min_{\vec{x}} f(\vec{x}) \\ \text{subject to} \quad \vec{c}_E(\vec{x}) = 0 \\ \vec{c}_I(\vec{x}) - \vec{s} = 0 \\ \vec{s} \geq 0 \end{aligned} \tag{17.24}$$

solve KKT, perturb from boundary

$$\begin{aligned} \nabla f - \lambda_E \cdot \nabla c_E - \lambda_I \cdot \nabla c_I = 0 \\ \vec{c}_E(\vec{x}) = 0 \\ \vec{c}_I(\vec{x}) - \vec{s} = 0 \\ \lambda_i s_i = \mu \end{aligned} \tag{17.25}$$

Newton step on system
decrease $\mu$
same as barrier

minimize

$$\min_{\vec{x},\vec{s}} f(x) - \mu \sum_i \log s_i$$
$$\text{subject to} \quad \vec{c}_E(\vec{x}) = 0$$
$$\vec{c}_I(\vec{x}) - \vec{s} = 0 \tag{17.26}$$

KKT for $s_i$

$$\mu \frac{1}{s_i} - \lambda_i = 0 \tag{17.27}$$

$$\lambda_i s_i = \mu \tag{17.28}$$

## 17.4 SELECTED REFERENCES

[Nocedal & Wright, 2006] Nocedal, Jorge, & Wright, Stephen J. (2006). *Numerical Optimization*. 2nd edn. New York: Springer.

Unusually clear coverage of a field full of unusually opaque books.

## 17.5 PROBLEMS

(17.1) Given a point $(x_0, y_0)$, analytically find the closest point on the line $y = ax + b$ by minimizing the distance $d^2 = (x_0 - x)^2 + (y_0 - y)^2$ subject to the constraint $y - ax - b = 0$.

(17.2) Consider a set of $N$ nodes that has each measured a quantity $x_i$. The goal is to find the best estimate $\bar{x}$ by minimizing

$$\min_{\bar{x}} \sum_{i=1}^{N} (\bar{x} - x_i)^2 \quad , \tag{17.29}$$

however each node $i$ can communicate only with nodes $j$ in its neighborhood $j \in \mathcal{N}(i)$. This can be handled by having each node obtain a local estimate $\bar{x}_i$, and introducing a consistency constraint $c_{ij} = \bar{x}_i - \bar{x}_j = 0 \; \forall \; j \in \mathcal{N}(i)$.

(*a*) What is the Lagrangian?

(*b*) Find an update rule for the estimates $\bar{x}_i$ by evaluating where the gradient of the Lagrangian vanishes.

(*c*) Find an update rule for the Lagrange multipliers by taking a Newton step on their constraints.

(17.3) What is the Newton step for the interior point KKT system?

(17.4) Solve a 1D spin glass (Problem 15.2) as a constrained optimization with relaxed spins.