3 Ordinary Differential and Difference Equations

3.1 LINEAR DIFFERENTIAL EQUATIONS

Change is the most interesting aspect of most systems, hence the central importance across disciplines of differential equations. An *ordinary differential equation* (*ODE*) is an equation (or system of equations) written in terms of an unknown function and its derivatives with respect to a single independent variable (such as time). Examples include the familiar equations of classical mechanics and electrical circuits. In the next chapter we will consider *partial differential equations* (*PDEs*), which have multiple independent variables (such as space, for example in fluid flow or electrodynamics). The subject of differential equations can appear to be quite tedious. In part it is: it is like learning spelling and grammar as a necessary prelude to the study of Shakespeare. And in part it isn't: there can be beautiful structure lurking behind what appear to be very simple differential equations. This chapter will concentrate on the canon of linear (or nearly linear) differential equations; after detouring through many other supporting topics the book will return to consider nonlinear differential equations in the closing chapter on time series.

The simplest differential equation can immediately be solved by integration

$$\frac{dy}{dt} = f(t) \Rightarrow dy = f(t)\, dt$$
$$\Rightarrow y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t)\, dt \tag{3.1}$$

(a point that is surprisingly often forgotten). The *order* of a differential equation is the highest derivative that occurs, and so the preceeding example is a first-order equation. If every term involves either the unknown function or its derivatives the equation is said to be *homogeneous*; if there is a term that depends on the independent variable alone (i.e., a forcing term) then the equation is *inhomogeneous*. If the unknown function does not appear within powers or more complicated functions, then the differential equation is linear, and can be written in terms of a linear operator $L_N(y)$ defined by

$$L_N(y) \equiv \frac{d^N y}{dt^N} + A_1(t)\frac{d^{N-1}y}{dt^{N-1}} + \cdots + A_{N-1}(t)\frac{dy}{dt} + A_N(t)y \quad . \tag{3.2}$$

There is no need for an A_0 coefficient because it can be eliminated by dividing all the other terms by it. $L_N(y) = f(t)$ is an inhomogeneous equation, and $L_N(y) = 0$ is the associated homogeneous equation.

Linear differential equations are particularly important, in part because they occur so

often (particularly in systems that are not strongly driven), and in part because general techniques exist for solving them (whether or not they really apply to what might be a nonlinear problem). Although this can be a bit like the proverbial drunk looking for lost change under a street lamp, it is sensible if it is the only illumination available.

The solution of an Nth-order linear differential equation will contain N unknown constants that are determined by *boundary conditions*. If it is an *initial-value problem*, the initial values of N independent functions of the variable and its derivatives are given (usually, $y(0)$, $dy/dt(0)$, ..., $d^{N-1}y/dt^{N-1}(0)$). For a *boundary-value problem*, boundary conditions are given at both the beginning and the end of an interval.

An N-order homogeneous equation $L_N(y) = 0$ will have N linearly independent solutions $u_1(t)$, $u_2(t)$, ..., $u_N(t)$. By superposition, an arbitrary linear combination of them will also be a solution:

$$y_g(t) = \sum_{n=1}^{N} C_n u_n(t) \quad . \tag{3.3}$$

This is the *general solution*; any solution of the homogeneous equation can be represented by an appropriate choice of the C_n's. If a *particular solution* of the inhomogeneous problem can be found ($L_N(y_p) = f(t)$), then the *complete solution* is $y(t) = y_g(t) + y_p(t)$. The general solution represents the transient response of the system to the boundary conditions, and the particular solution is the result of the forcing of the system by the inhomogeneous term.

The simplest linear differential equation has constant coefficients:

$$\frac{d^N y}{dt^N} + A_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + A_{N-1} \frac{dy}{dt} + A_N y = f(t) \quad . \tag{3.4}$$

An important technique for solving differential equations is to guess the functional form of a solution (called an *ansatz*, or trial answer), substitute it in, and then see if the free parameters can be adjusted to make the solution work. Because the solution of a differential equation is unique as long as the functions defining it are reasonably smooth and bounded [Coddington & Levinson, 1984], if you find *a* solution then that is *the* solution. Guessing $y = e^{rt}$ for the solution of the homogeneous part of equation (3.4)

$$L_N(e^{rt}) = e^{rt} \left(r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N \right) = 0 \tag{3.5}$$

leads to the *characteristic equation*

$$r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = 0 \quad . \tag{3.6}$$

This Nth-order polynomial has N roots. The real part of the roots represent exponentially growing or decaying solutions, and the complex part oscillatory behavior. If all of the roots are distinct:

$$r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = (r - r_1)(r - r_2) \cdots (r - r_N) \tag{3.7}$$

then the general solution is

$$y_g = \sum_{n=1}^{N} C_n e^{r_n t} \quad . \tag{3.8}$$

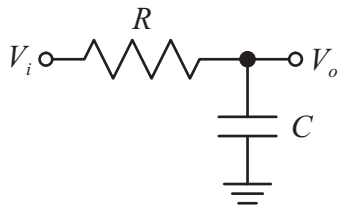

Figure 3.1. An RC circuit.

This gives the N linearly independent solutions required for a general solution. However, if a root has a higher multiplicity

$$r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = (r - r_1)^M (r - r_{M+1}) \cdots (r - r_N) \tag{3.9}$$

then this will provide fewer than N solutions. The missing solutions can be found by differentiating:

$$\begin{aligned} \frac{d}{dr} L_N(e^{rt}) &= \frac{d}{dr}\, e^{rt} \prod_{n=1}^{N} (r - r_n) \\ &= te^{rt} \prod_{n=1}^{N} (r - r_n) + e^{rt} \frac{d}{dr} \prod_{n=1}^{N} (r - r_n) \quad . \end{aligned} \tag{3.10}$$

The first term in equation (3.10) will always vanish at each of the roots, but the second term need not. For a root of multiplicity one,

$$\frac{d}{dr}\,(r - r_1) \prod_{n \neq 1} (r - r_n) = \prod_{n \neq 1} (r - r_n) + (r - r_1)\, \frac{d}{dr} \prod_{n \neq 1} (r - r_n) \tag{3.11}$$

the first term in the derivative of the characteristic equation will not vanish at the root. But for a double root

$$\frac{d}{dr}\,(r - r_1)^2 \prod_{n \neq 1} (r - r_n) = 2(r - r_1) \prod_{n \neq 1} (r - r_n) + (r - r_1)\, \frac{d}{dr} \prod_{n \neq 1} (r - r_n) \tag{3.12}$$

both terms will. Therefore

$$\frac{d}{dr} L_N(e^{rt}) = L_N\left(\frac{de^{rt}}{dr}\right) = L_N(te^{rt}) = 0 \tag{3.13}$$

and we've found that te^{rt} is another solution associated with the root. Likewise, for a triple root

$$\frac{d^2}{dr^2} L_N(e^{rt}) = L_N\left(\frac{d^2}{dr^2} e^{rt}\right) = L_N(t^2 e^{rt}) = 0 \quad , \tag{3.14}$$

and for an Mth-order root the M linearly independent solutions are

$$y = (C_1 + C_2 t + C_3 t^2 + \cdots + C_M t^{M-1}) e^{r_1 t} \quad . \tag{3.15}$$

As a simple example of a linear constant-coefficient differential equation, consider a circuit consisting of a resistor and a capacitor (Figure 3.1). The current into the node

from the resistor is $(V_i - V_o)/R$, and the current out of the node into the capacitor is $C\dot{V}_o$, and so the governing equation for this circuit is

$$C\dot{V}_o = \frac{V_i - V_o}{R} \tag{3.16}$$

or

$$RC\dot{V}_o + V_o = V_i \quad . \tag{3.17}$$

The characteristic equation gives

$$RC \cdot r + 1 = 0 \;\Rightarrow\; r = \frac{-1}{RC} \;\Rightarrow\; V_o = Ae^{-t/RC} \quad . \tag{3.18}$$

The undriven response of the circuit is to exponentially discharge the capacitor. Now, let's assume periodic forcing $V_i = \exp(i\omega t)$ and look for a particular solution at this frequency. The voltage in the circuit is of course a real number; by representing it as a complex number we can simultaneously keep track of both phase components (sin and cos). Plugging in the ansatz $V_o = A\exp(i\omega t)$ gives

$$RCAi\omega + A = 1 \;\Rightarrow\; A = \frac{1}{1 + i\omega RC} \quad . \tag{3.19}$$

At low frequencies the output is equal to the input; at high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by 90°. Problem 3.1 covers the important example of a damped, driven harmonic oscillator.

This completes (more-or-less) everything that there is to know about solving linear differential equations. The theory is simple and useful. The situation is very different for nonlinear differential equations, where amidst a sea of insoluble problems live special tricks for some tractable equations, approximation methods based on some nearby exactly soluble problems, and qualitative insights into the behavior of classes of solutions. Because of this, the study of nonlinear differential equations requires either a lot of specialized attention or else numerical methods.

Another extension of this basic theory is to coupled systems of equations. Once again, little general can be said about nonlinear systems, but for the case of linear couplings it is possible to find exact solutions. The next section looks at this for the important case of coupled oscillators.

3.2 SYSTEMS OF DIFFERENTIAL EQUATIONS AND NORMAL MODES

The Nth-order linear differential equation (3.4) can be written as a first-order equation for an N-dimensional vector

$$\frac{d}{dt}\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix} = \tag{3.20}$$

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -A_N & -A_{N-1} & -A_{N-2} & \cdots & -A_1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix} \quad .$$

This transformation does not make the problem any simpler (it can be solved by *diagonalizing* the matrix [Strang, 2016], which requires solving exactly the same characteristic equation), but it can be convenient to simplify notation by using a vector first-order equation [Gershenfeld *et al*., 1983].

This is a simple example of a system of differential equations. Such systems also arise whenever there are interactions; an important special case is an unforced, undamped system of masses with coordinates $(y_1, y_2, \ldots, y_N) \equiv \vec{y}$ that have a restoring force that is an arbitrary linear combination of their positions. The corresponding vector equation is

$$\frac{d^2\vec{y}}{dt^2} + \mathbf{A} \cdot \vec{y} = 0 \quad . \tag{3.21}$$

If the coupling matrix $\mathbf{A}$ is diagonal ($A_{ij} = 0$ for $i \neq j$) then the oscillators will be independent, but if it isn't then they won't. Let's look for a new set of variables $\vec{z} \equiv \mathbf{M}^{-1} \cdot \vec{y}$, defined by an unknown transformation $\mathbf{M}$, for which these equations decouple:

$$\frac{d^2\vec{z}}{dt^2} + \mathbf{D} \cdot \vec{z} = 0 \quad , \tag{3.22}$$

where $\mathbf{D}$ is a diagonal matrix. The required transformation $\mathbf{M}$ can be found by changing variables:

$$\begin{aligned} \frac{d^2\vec{y}}{dt^2} + \mathbf{A} \cdot \vec{y} &= 0 \\ \mathbf{M} \cdot \frac{d^2\vec{z}}{dt^2} + \mathbf{A} \cdot \mathbf{M} \cdot \vec{z} &= 0 \\ \frac{d^2\vec{z}}{dt^2} + \mathbf{M}^{-1} \cdot \mathbf{A} \cdot \mathbf{M} \cdot \vec{z} &= 0 \\ \frac{d^2\vec{z}}{dt^2} + \mathbf{D} \cdot \vec{z} &= 0 \end{aligned} \tag{3.23}$$

and so $\mathbf{M}^{-1} \cdot \mathbf{A} \cdot \mathbf{M} = \mathbf{D}$ or $\mathbf{A} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{D}$. If the nth *eigenvector* of $\mathbf{A}$ is $\vec{v}_n$, with corresponding *eigenvalue* $\mathbf{A} \cdot \vec{v}_n = \vec{v}_n \lambda_n$, then if $\mathbf{M} = [\vec{v}_1 \cdots \vec{v}_N]$ has the eigenvectors for columns

$$\begin{aligned} \mathbf{A} \cdot \mathbf{M} &= \mathbf{A} \cdot [\vec{v}_1 \cdots \vec{v}_N] \\ &= [\vec{v}_1 \cdots \vec{v}_N] \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \\ &= \mathbf{M} \cdot \mathbf{D} \end{aligned} \tag{3.24}$$

it will diagonalize $\mathbf{M}$ and hence satisfy equation (3.23). The new variables here are called *normal modes* [Goldstein *et al*., 2001; Scheck, 2010] and behave exactly like independent oscillators. There will be as many normal modes as there are degrees of freedom, unless

there are fewer distinct eigenvectors because of degenerate eigenvalues. Problem 3.2 finds the normal modes for a simple system.

3.3 LAPLACE TRANSFORMS

Using the characteristic equation to solve a differential equation requires separate steps to find the general solution, search for a particular solution, and solve for the coefficients to match the boundary conditions. *Laplace transforms* provide a convenient alternative, turning many differential equations into an algebraic problem and giving the complete solution in a single step.

The *one-sided Laplace transform* of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad . \tag{3.25}$$

If the integral extended from $-\infty$ to ∞ this would be the *two-sided* Laplace transform. The one-sided transform explicitly includes the initial conditions of the system at $t = 0$, and for this reason we will use it; the two-sided transform is used for steady-state problems for which the initial conditions do not matter.

The Laplace transform is a generalization of the Fourier transform to an arbitrary complex argument. Its usefulness for differential equations comes from recognizing that differentiation just multiplies that Laplace transfrom by s:

$$\begin{aligned} \mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= \int_0^\infty e^{-st}\frac{df(t)}{dt}\, dt \\ &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\ &= sF(s) - f(0) \quad , \end{aligned} \tag{3.26}$$

where the second step follows by integrating by parts

$$\int_A^B u\, dv = uv\Big|_A^B - \int_A^B v\, du \quad . \tag{3.27}$$

Similarly, for second derivatives

$$\begin{aligned} \mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} &= \int_0^\infty e^{-st}\frac{d^2 f(t)}{dt^2}\, dt \\ &= e^{-st}\frac{df(t)}{dt}\Big|_0^\infty + s\int_0^\infty e^{-st}\frac{df(t)}{dt}\, dt \\ &= s^2 F(s) - sf(0) - \frac{df(0)}{dt} \quad , \end{aligned} \tag{3.28}$$

and by induction for higher derivatives

$$\begin{aligned} \mathcal{L}\left\{\frac{d^N f(t)}{dt^N}\right\} &= s^N F(s) - s^{N-1} f(0) - s^{N-2}\frac{df(0)}{dt} \\ &\quad - s^{N-3}\frac{d^2 f(0)}{dt^2} - \cdots - \frac{d^{N-1} f(0)}{dt^{N-1}} \end{aligned} \tag{3.29}$$

(see [Hildebrand, 1976] for the continuity requirements assumed by these results). There is a corresponding relationship for integrals:

$$\begin{aligned}\mathcal{L}\left\{\int_0^t f(u)\,du\right\} &= \int_0^\infty e^{-st}\int_0^t f(u)\,du\,dt \\ &= -\frac{e^{-st}}{s}\int_0^t f(u)\,du\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt \\ &= \frac{1}{s}F(s) \quad . \end{aligned} \tag{3.30}$$

The Laplace transform turns differential and integral equations into algebraic equations. Without proof, Table 3.1 gives a number of other Laplace transform pairs (much longer tables are available in any mathematics handbook). Where needed it is assumed that the arguments vanish for $t < 0$.

In order to solve a differential equation by using Laplace transforms, the steps are

1. Laplace transform the differential equation into an algebraic equation, including the initial conditions.
2. Solve this new equation for the unknown function in terms of the transform variable s.
3. Find the inverse transform.

There is not an automatic way to invert a Laplace transform. The easiest approach, when it works, is to look up the inverse in a table of Laplace transforms. Many more problems can be handled by first doing a *partial fraction expansion* to simplify them. The transform $F(s)$ of a constant-coefficient linear differential equation will be the ratio of two polynomials

$$F(s) = \frac{N(s)}{D(s)} \quad , \tag{3.31}$$

called a *rational function*. The roots of the numerator $N(s)$ are called *zeros*, and the roots of the denominator $D(s)$ are called *poles*. Let the poles be $\{p_i\}_{i=1}^N$. The *partial fraction expansion* of an $F(s)$ with distinct poles and a numerator of lower degree than the denominator is [Pearson, 1990]

$$F(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i} \tag{3.32}$$

where

$$A_i = \lim_{s \to p_i} F(s)(s - p_i) \quad . \tag{3.33}$$

The definition of the coefficients can be verified by substitution:

$$F(s)(s - p_i)\big|_{s=p_i} = A_i + \sum_{j \neq i}^{N} A_j \frac{s - p_i}{s - p_j}\bigg|_{s=p_i} = A_i \quad . \tag{3.34}$$

For repeated poles p_n with a multiplicity n, the terms in the partial fraction expansion are of the form

$$F(s) = \frac{A_n}{(s - p_n)^n} + \frac{A_{n-1}}{(s - p_n)^{n-1}} + \cdots + \frac{A_1}{s - p_n} \quad , \tag{3.35}$$

Table 3.1. *Selected Laplace transforms.*

$$\mathcal{L}\left\{t^N f(t)\right\} = (-1)^N \frac{d^N F(s)}{ds^N}$$

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

$$\mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\} \equiv \mathcal{L}\{f * g\} = \text{convolution} = F(s)G(s)$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$\mathcal{L}\left\{\frac{1}{a-b}(-e^{-at}+e^{-bt})\right\} = \frac{1}{(s+a)(s+b)}$$

$$\mathcal{L}\left\{\frac{1}{a-b}(ae^{-at}-be^{-bt})\right\} = \frac{s}{(s+a)(s+b)}$$

$$\mathcal{L}\left\{\frac{e^{-at}}{(a-b)(a-c)} - \frac{e^{-bt}}{(a-b)(b-c)} - \frac{e^{-ct}}{(a-c)(c-b)}\right\} = \frac{1}{(s+a)(s+b)(s+c)}$$

$$\mathcal{L}\left\{-\frac{ae^{-at}}{(a-b)(a-c)} + \frac{be^{-bt}}{(a-b)(b-c)} + \frac{ce^{-ct}}{(a-c)(c-b)}\right\} = \frac{s}{(s+a)(s+b)(s+c)}$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\mathcal{L}\left\{t^N\right\} = \frac{N!}{s^{N+1}}$$

$$\mathcal{L}\left\{\frac{t^{N-1}e^{-at}}{(N-1)!}\right\} = \frac{1}{(s+a)^N}$$

with

$$A_n = \lim_{s\to p_n} F(s)(s-p_n)^n \quad . \tag{3.36}$$

Finally, some problems can be solved by doing a complex contour integral to invert equation (3.25) [Saff & Snider, 2003].

Let's return to the simple example of an RC circuit:

$$RC\dot{V}_o(t) + V_o(t) = V_i(t)$$

$$RC[sV_o(s) - V_o(0)] + V_o(s) = V_i(s)$$

$$V_o(s) = \frac{V_i(s)}{1 + sRC} + \frac{RCV_o(0)}{1 + sRC}$$

$$V_o(t) = (RC)^{-1} e^{-t/RC} * V_i(t) + V_o(0) e^{-t/RC} \tag{3.37}$$

(from Table 3.1). We see immediately that the output is the sum of the transient decay of the initial state and the convolution of the exponential decay with the input. If as before we take the forcing to be $\exp(i\omega t)$,

$$V_o(s) = \frac{1}{1 + sRC} \frac{1}{s - i\omega} + \frac{RCV_o(0)}{1 + sRC}$$

$$V_o(t) = \frac{1}{1 + i\omega RC} \left(e^{i\omega t} - e^{-t/RC} \right) + V_o(0) e^{-t/RC} \tag{3.38}$$

(using Table 3.1). We did not see the second part of the first term before because we did not worry about satisfying the initial conditions; here it arises naturally from the use of the Laplace transform.

The output $y(t)$ from an arbitrary causal linear time-invariant system (one in which the future is uniquely determined by the past, and the equations do not change over time, such as a constant-coefficient linear differential equation) is related to the input $x(t)$ by convolution with respect to the impulse response $h(t)$ (the response of the system to a delta-function input)

$$y(t) = \int_0^t h(\tau) x(t - \tau) \, d\tau \quad . \tag{3.39}$$

Since convolution in the time domain equals multiplication of Laplace transforms,

$$H(s) = \frac{Y(s)}{X(s)} \quad . \tag{3.40}$$

$H(s)$, the Laplace transform of the impulse response, is called the system *transfer function* and is equal to the ratio of the Laplace transforms of the input and the output.

A general transfer function for a finite-dimensional linear system will be the ratio of two polynomials. Since the polynomials can be constructed from knowledge of the roots, the location of the poles and zeros completely characterizes the response of the system. A great deal therefore can be learned about a system from the placement of its poles and zeros. Most importantly, for the system to be globally stable, all of the poles must lie in the left half–plane $\mathrm{Re}(p_i) < 0$ (recall that $\mathcal{L}\{\exp(at)\} = 1/(s - a)$, and so if $\mathrm{Re}(a) > 0$ then the solution will diverge). Similarly, poles off of the real axis are associated with oscillatory solutions.

If the input is $x = \exp(i\omega t)$, then the asymptotic output is

$$\begin{aligned} y(t) &= \lim_{t \to \infty} \int_0^t h(\tau) x(t - \tau) \, d\tau \\ &= \lim_{t \to \infty} \int_0^t h(\tau) e^{i\omega(t - \tau)} \, d\tau \\ &= \lim_{t \to \infty} e^{i\omega t} \int_0^t h(\tau) e^{-i\omega\tau} \, d\tau \\ &= e^{i\omega t} H(i\omega) \quad . \end{aligned} \tag{3.41}$$

The steady-state output (after any initial transients) is equal to the input multiplied by the transfer function evaluated at $i\omega$.

3.4 PERTURBATION EXPANSIONS

One is much more likely to encounter a differential equation that is not analytically soluble than one that is. If the problem is related to one that does have an analytical solution, it can be possible to find an approximate solution by an expansion around the known one. The key is to recognize a small parameter ϵ in the problem that in the limit $\epsilon \to 0$ reduces to the known problem. Equally useful is a very large parameter, since its inverse can be used to develop an approximation.

For example, consider a harmonic oscillator with a weak nonlinear dissipation

$$\ddot{x} + x + \epsilon \dot{x}^2 = 0 \quad . \tag{3.42}$$

Clearly, when ϵ is small the solution will not be too far from simple harmonic motion. As an ansatz we will expand the solution in powers of the small parameter:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \mathcal{O}(\epsilon^3) \quad . \tag{3.43}$$

The notation $\mathcal{O}(\epsilon^3)$ stands for terms of order ϵ^3 and higher. Now, plug this in and collect terms based on their order. All of the terms with the same order of ϵ must satisfy the differential equation independently of the others, because their relationship will change if ϵ is arbitrarily varied:

$$\ddot{x}_0 + x_0 + \epsilon \dot{x}_0^2 + \epsilon \ddot{x}_1 + \epsilon x_1 + \epsilon^3 \dot{x}_1^2 + \cdots = 0$$

$$\mathcal{O}(\epsilon^0) : \ddot{x}_0 + x_0 = 0$$

$$\mathcal{O}(\epsilon^1) : \ddot{x}_1 + x_1 + \dot{x}_0^2 = 0 \tag{3.44}$$

and so forth. The lowest-order equation is just that for the unperturbed oscillator, and the higher-order equations give corrections. This is a hierarchy that can be solved in order, first finding x_0, then using it to find x_1, and on up to the desired order of approximation. If the initial conditions are chosen so that $x_0 = \exp(it)$, then this gives for x_1

$$\ddot{x}_1 + x_1 = e^{i2t} \quad . \tag{3.45}$$

The homogeneous solution is $A\exp(it) + B\exp(-it)$, and a particular solution can be found by plugging in $x_1 = C\exp(i2t)$, which gives $C = -1/3$. The nonlinearity has to first order introduced a harmonic into the system, a familiar feature of strongly driven systems.

Perturbation approximations are useful if there is a natural notion of a small deviation from a soluble problem. Another important kind of approximation is an *asymptotic expansion*, which is a series expansion that applies for very large arguments (such as $t \to \infty$). See, for example, [Pearson, 1990].

3.5 DISCRETE TIME EQUATIONS

It is a common and usually reasonable approximation to consider physical systems to be continuous, and so differential equations apply. But this is not the case for digital systems, which usually are discretized in time (there is a system clock). Conveniently, the theory of discrete time equations is essentially identical to that for differential equations.

Take $y(k)$ to be a series defined only at integer values of the time k (replace k with $k\Delta t$ throughout this section if the time step is not an integer). An N-th order linear constant coefficient difference equation for $y(k)$ is

$$L_N(y) \equiv y(k) + A_1 y(k-1) + \cdots + A_{N-1} y(k-N+1) + A_N y(k-N) = f(k) \quad . \tag{3.46}$$

Substituting the ansatz $y(k) = r^k$ into the homogeneous equation ($f(k) = 0$) and cancelling the r^k terms gives the same characteristic equation we saw for differential equations,

$$r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = 0 \quad . \tag{3.47}$$

Once again, if all of the roots of this N-th order polynomial are distinct

$$r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = (r - r_1)(r - r_2) \cdots (r - r_N) \tag{3.48}$$

then the general solution is

$$y_g = \sum_{n=1}^{N} C_n r_n^k \quad , \tag{3.49}$$

where the C_n's are unknown coefficients determined by the boundary equations. A complete solution is made up of this general solution plus a particular solution to the inhomogeneous problem.

If some of the roots of the characteristic polynomial are repeated,

$$r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = (r - r_1)^M (r - r_{M+1}) \cdots (r - r_N) \quad , \tag{3.50}$$

then recognizing that the operations of differentiation and shifting in time can be interchanged, and repeating the argument preceeding equation (3.15), we see that the extra solutions associated with the repeated roots are

$$\frac{d}{dr} L_N(r^k) = L_N\left(\frac{dr^k}{dr}\right) = L_N(k r^{k-1}) = 0 \quad ,$$

$$\frac{d^2}{dr^2} L_N(r^k) = L_N\left(\frac{d^2}{dr^2} r^k\right) = L_N(k(k-1) r^{k-2}) = 0 \quad , \tag{3.51}$$

and so forth.

3.6 z-TRANSFORMS

For a series $y(k)$, the *one-sided z-transform* is defined by

$$\mathcal{Z}\{y(k)\} \equiv Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k} \quad . \tag{3.52}$$

Table 3.2. *Selected z-transforms.*

$$\mathcal{Z}\{kf(k)\} = -z\frac{dF(z)}{dz}$$
$$\mathcal{Z}\left\{a^k f(k)\right\} = F(z/a)$$
$$\mathcal{Z}\left\{\sum_{n=0}^{k} f(k-n)g(n)\right\} \equiv \mathcal{Z}\{f * g\} = F(z)G(z)$$
$$\mathcal{Z}\{1\} = \frac{z}{z-1}$$
$$\mathcal{Z}\left\{a^k\right\} = \frac{z}{z-a}$$
$$\mathcal{Z}\{\delta(k-n)\} = z^{-n}$$

As with the Laplace transform, a two-sided transform can also be defined for signals that start at $-\infty$, extending the discrete Fourier transform to the complex plane.

Just as differentiation is the most important property of Laplace transforms, time-shifting is the most important property for z-transforms. Consider a series delayed by one time step:

$$\begin{aligned}\mathcal{Z}\{y(k-1)\} &= \sum_{k=0}^{\infty} y(k-1)z^{-k} \\ &= \sum_{k'=-1}^{\infty} y(k')z^{-k'}z^{-1} \quad (k' = k-1) \\ &= z^{-1}\sum_{k'=0}^{\infty} y(k')z^{-k'} + z^{-1}y(-1)z \\ &= z^{-1}Y(z) + y(-1) \ . \end{aligned} \tag{3.53}$$

Similarly, for a delay of two,

$$\begin{aligned}\mathcal{Z}\{y(k-2)\} &= \sum_{k=0}^{\infty} y(k-2)z^{-k} \\ &= \sum_{k'=-2}^{\infty} y(k')z^{-k'}z^{-2} \quad (k' = k-2) \\ &= z^{-2}\sum_{k'=0}^{\infty} y(k')z^{-k'} + z^{-2}y(-1)z + z^{-2}y(-2)z^2 \\ &= z^{-2}Y(z) + z^{-1}y(-1) + y(-2) \ , \end{aligned} \tag{3.54}$$

and so forth for longer delays. The z-transform turns a difference equation into an algebraic equation. Without proof, Table 3.2 gives a few other z-transforms (where $f(k) = 0$ for $k < 0$); see the references for many more.

Many of the properties of the Laplace transform carry over to the z-transform. If the input forcing to a constant-coefficient linear difference equation is a unit impulse $x(k) = \delta(k)$ ($\delta(0) = 1$, $\delta(k \neq 0) = 0$), then the solution $y(k) = h(k)$ is defined to be the impulse response of the system. The output for an arbitrary input is given by the

convolution with the impulse response

$$y(k) = \sum_{n=0}^{k} h(n)x(k-n) \tag{3.55}$$

and in the z domain the transfer function (the z-transform of the impulse response) is the ratio of the transforms of the input and the output

$$H(z) = \frac{Y(z)}{X(z)} \quad . \tag{3.56}$$

The stability of a system is determined by the location of the poles of the transfer function; for it to be stable for any bounded input signal the poles must have a complex magnitude of less than 1 (recall that $\mathcal{Z}\{a^k\} = z/(z-a)$, which has a pole at a, so $|a|$ must be less than 1 for a^k to remain bounded).

The frequency response can be found with a calculation similar to the continuous time case. If the input is $x(k) = \exp(i\omega\delta_t k)$, then the asymptotic output is

$$\begin{aligned} y(k) &= \lim_{k\to\infty} \sum_{n=0}^{k} h(n)x(k-n) \\ &= \lim_{k\to\infty} \sum_{n=0}^{k} h(n)e^{i\omega\delta_t(k-n)} \\ &= \lim_{k\to\infty} e^{i\omega\delta_t k} \sum_{n=0}^{k} h(n)e^{-i\omega\delta_t n} \\ &= e^{i\omega\delta_t k} H(e^{i\omega\delta_t}) \quad . \end{aligned} \tag{3.57}$$

Now it is multiplied by the transfer function evaluated at $\exp(i\omega\delta_t)$. Problem 3.3 looks at the solution for, and frequency response of, a simple digital filter.

3.7 SELECTED REFERENCES

[Strang, 2016] Strang, Gilbert. (2016). *Introduction to Linear Algebra*. 5th edn. Wellesley-Cambridge Press.

A very readable introduction to applied mathematics, including linear algebra and differential equations.

[Zwillinger, 1997] Zwillinger, Daniel. (1997). *Handbook of Differential Equations*. 3rd edn. New York: Academic Press.

A good meta-index to most known techniques (analytical and numerical) for studying differential equations.

[Kamen, 1990] Kamen, Edward W. (1990). *Introduction to Signals and Systems*. 2nd edn. New York, NY: Macmillan.

Because of its importance in engineering practice, there are many books with a title similar to this that cover continuous and discrete time systems (Laplace and z-transforms). Kamen's book provides a nice balance between rigor and insight.

[Oppenheim & Schafer, 2009] Oppenheim, A.V., & Schafer, R.W. (2009). *Discrete-Time Signal Processing*. 3rd edn. Englewood Cliffs, NJ: Prentice Hall.

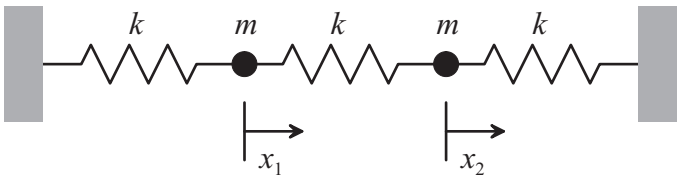

Figure 3.2. Two coupled harmonic oscillators.

A definitive reference for one of the most important applications of the theory of discrete time systems, digital signal processing.

3.8 PROBLEMS

(3.1) Consider the motion of a damped, driven harmonic oscillator (such as a mass on a spring, a ball in a well, or a pendulum making small motions):

$$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t} \quad . \tag{3.58}$$

(*a*) Under what conditions will the governing equations for small displacements of a particle around an arbitrary 1D potential minimum be simple undamped harmonic motion?

(*b*) Find the solution to the homogeneous equation, and comment on the possible cases. How does the amplitude depend on the frequency?

(*c*) Find a particular solution to the inhomogeneous problem by assuming a response at the driving frequency, and plot its magnitude and phase as a function of the driving frequency for $m = k = 1, \gamma = 0.1$.

(*d*) For a driven oscillator the Q or *Quality factor* is defined as the ratio of the center frequency to the width of the curve of the average energy (kinetic + potential) in the oscillator versus the driving frequency (the width is defined by the places where the curve falls to half its maximum value). For an undriven oscillator the Q is defined to be the ratio of the energy in the oscillator to the energy lost per radian (one cycle is 2π radians). Show that these two definitions are equal, assuming that the damping is small. How long does it take the amplitude of a 100 Hz oscillator with a Q of 10^9 to decay by $1/e$?

(*e*) Now find the solution to equation (3.58) by using Laplace transforms. Take the initial condition as $x(0) = \dot{x}(0) = 0$.

(*f*) For an arbitrary potential minimum, work out the form of the lowest-order correction to simple undamped unforced harmonic motion.

(3.2) Explicitly solve (and try to simplify) the system of differential equations for two coupled harmonic oscillators (see Figure 3.2; don't worry about the initial transient), and then find the normal modes by matrix diagonalization.

(3.3) A common simple digital filter used for smoothing a signal is

$$y(k) = \alpha y(k-1) + (1-\alpha)x(k) \quad , \tag{3.59}$$

where α is a parameter that determines the response of the filter. Use z-transforms to solve for $y(k)$ as a function of $x(k)$ (assume $y(k < 0) = 0$). What is the amplitude of the frequency response?