21 Control Theory

This chapter more appropriately belongs in a book titled *The Nature of Mathematical Manipulation*, but it is here because the goal of modeling is rarely just the edification of the modeler: usually the intent is to act on the results of the model to achieve a desired goal. *Control theory* seeks to vary inputs to a system to produce desired outputs, a process that can range from years to femtoseconds [Rabitz *et al.*, 2000].

oven on-off proportional PID

state-space

Chapter 5 equations of motion Chapter 7 numerically solve

inverted pendulum

Problem 5.3

$$l\ddot{\theta} + (g + \ddot{z})\,\theta = 0 \quad . \tag{21.1}$$

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (g+\ddot{z})/l & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \tag{21.2}$$

parametric

discrete time

$$\begin{aligned} \vec{x}_{n+1} &= \mathbf{A}\vec{x}_n + \mathbf{B}\vec{u}_n \\ \vec{y}_n &= \mathbf{C}\vec{x}_n + \mathbf{D}\vec{u}_n \end{aligned} \tag{21.3}$$

solve

$$\begin{aligned} \vec{x}_1 &= \mathbf{A}\vec{x}_0 + \mathbf{B}\vec{u}_0 \\ \vec{x}_2 &= \mathbf{A}\vec{x}_1 + \mathbf{B}\vec{u}_1 \\ &= \mathbf{A}\left[\mathbf{A}\vec{x}_0 + \mathbf{B}\vec{u}_0\right] + \mathbf{B}\vec{u}_1 \\ &= \mathbf{A}^2\vec{x}_0 + \mathbf{A}\mathbf{B}\vec{u}_0 + \mathbf{B}\vec{u}_1 \\ &\vdots \\ \vec{x}_n &= \mathbf{A}^n\vec{x}_0 + \mathbf{B}\vec{u}_{n-1} + \mathbf{A}\mathbf{B}\vec{u}_{n-2} + \cdots + \mathbf{A}^{n-1}\mathbf{B}\vec{u}_0 \end{aligned} \tag{21.4}$$

stability eigenvalues magnitude $>, < 1$

continuous time

$$\dot{\vec{x}} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$$
$$\vec{y} = \mathbf{C}\vec{x} + \mathbf{D}\vec{u} \tag{21.5}$$

matrix exponential

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots \tag{21.6}$$

solution

$$\vec{x}(t) = e^{\mathbf{A}t}\vec{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\vec{u}(\tau)\,d\tau \tag{21.7}$$

stability eigenvalues real part $>, < 0$
Laplace transform

$$s\vec{X}(s) = \mathbf{A}\vec{X}(s) + \mathbf{B}\vec{U}(s)$$
$$\vec{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\vec{U}(s) \tag{21.8}$$

$$\vec{Y}(s) = \left(\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)\vec{U}(s) \tag{21.9}$$

$\vec{Y}(s) = \mathbf{G}(s)\vec{U}(s)$

$$\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} \tag{21.10}$$

transfer function poles = system eigenvalues
controllability
discrete time

$$\vec{U}_{n-1} \equiv \begin{bmatrix} \vec{u}_{n-1} \\ \vdots \\ \vec{u}_0 \end{bmatrix} \tag{21.11}$$

$$\mathbf{P}_{n-1} \equiv \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \tag{21.12}$$

$$\vec{x}_n = \mathbf{A}^n\vec{x}_0 + \mathbf{P}_{n-1}\vec{U}_{n-1} \tag{21.13}$$

P full row rank of $\vec{x} = N$
saturates at $n = N$ from Cayley-Hamilton (every matrix satisfies its own characteristic equation)
divide by $\Delta(\mathbf{A})$:

$$P(\mathbf{A}) = Q(\mathbf{A})\Delta(\mathbf{A}) + R(\mathbf{A}) \tag{21.14}$$

continuous time:

$$
\begin{aligned}
\vec{x}(t) - e^{\mathbf{A}t}\vec{x}_0 &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\vec{u}(\tau)\, d\tau \\
&= \int_0^t \left[\mathbf{I} + \mathbf{A}(t-\tau) + \cdots + \frac{1}{(N-1)!}\mathbf{A}^{N-1}(t-\tau)^{N-1}\right]\mathbf{B}\vec{u}(\tau)\, d\tau \\
&= \sum_{n=0}^{N-1}\mathbf{A}^n\mathbf{B}\int_0^t \frac{1}{n!}(t-\tau)^n\vec{u}(\tau)\, d\tau \\
&\equiv \sum_{n=0}^{N-1}\mathbf{A}^n\mathbf{B}\vec{v}_n \\
&= \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{N-2}\mathbf{B} & \mathbf{A}^{N-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\vec{v}_0 \\ \vec{v}_1 \\ \vdots \\ \vec{v}_{N-1}\end{bmatrix}
\end{aligned} \tag{21.15}
$$

observability

$$
\vec{y}_n = \mathbf{CA}^n\vec{x}_0 + \mathbf{CB}\vec{u}_{n-1} + \mathbf{CAB}\vec{u}_{n-2} + \cdots + \mathbf{CA}^{n-1}\mathbf{B}\vec{u}_0 + \mathbf{D}\vec{u}_n \tag{21.16}
$$

$$
\mathbf{CA}^n\vec{x}_0 = \vec{y}_n - \mathbf{CB}\vec{u}_{n-1} + \mathbf{CAB}\vec{u}_{n-2} + \cdots + \mathbf{CA}^{n-1}\mathbf{B}\vec{u}_0 + \mathbf{D}\vec{u}_n \tag{21.17}
$$

know right hand side

$$
\mathbf{Q}_{n-1} \equiv \begin{bmatrix}\mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1}\end{bmatrix} \tag{21.18}
$$

full column rank N
$\vec{u} = 0$:

$$
\begin{aligned}
\vec{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\vec{x}_0 \\
&= \mathbf{C}\left[\mathbf{I} + \cdots + \frac{1}{(N-1)!}\mathbf{A}^{N-1}t^{N-1}\right]\vec{x}_0 \\
&= \sum_{n=0}^{N-1}\mathbf{CA}^n\frac{1}{n!}t^n\vec{x}_0 \\
&= \begin{bmatrix}\mathbf{C} & \mathbf{CA} & \cdots & \mathbf{CA}^{N-2} & \mathbf{CA}^{N-1}\end{bmatrix}\begin{bmatrix}\vec{x}_0 \\ t\vec{x}_0 \\ \vdots \\ \frac{1}{(N-1)!}t^{N-1}\vec{x}_0\end{bmatrix}
\end{aligned} \tag{21.19}
$$

output feedback
state variable feedback
feedback gain matrix **K**

$$\vec{u} = \vec{v} - \mathbf{K}\vec{x} \tag{21.20}$$

$$\dot{\vec{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\vec{x} + \mathbf{B}\vec{v} \tag{21.21}$$

$$|\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = 0 \tag{21.22}$$

expand, equate to desired characteristic polynomial, match orders λ
pole-zero placement, cancellation
more complex for multiple inputs

$$(\mathbf{A} - \mathbf{B}\mathbf{K})\,\vec{\varphi}_i = \lambda_i\vec{\varphi}_i \tag{21.23}$$

$$[(\lambda_i\mathbf{I} - \mathbf{A})\ \ \mathbf{B}]\begin{bmatrix} \vec{\varphi}_i \\ \mathbf{K}\vec{\varphi}_i \end{bmatrix} = \vec{0} \tag{21.24}$$

pick $\{\lambda_i\}$, find basis null space, solve for **K**
Lyapunov
stability
$v = \vec{x}^T\mathbf{P}\vec{x}, \mathbf{P} < 0$

$$\begin{aligned}\dot{v} &= \dot{\vec{x}}^T\mathbf{P}\vec{x} + \vec{x}^T\mathbf{P}\dot{\vec{x}} \\ &= \vec{x}^T\mathbf{A}^T\mathbf{P}\vec{x} + \vec{x}^T\mathbf{P}\mathbf{A}\vec{x} \\ &= \vec{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\vec{x}\end{aligned} \tag{21.25}$$

v decreasing:
$\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right) < 0$
LMI, feasibility
$\dot{\vec{x}} = \mathbf{A}\vec{x}$
$\vec{u} = \mathbf{K}\vec{x}$

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{K}) < 0 \tag{21.26}$$

convert to LMI

$$\mathbf{Q}(\mathbf{A} + \mathbf{B}\mathbf{K})^T + (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{Q} < 0 \tag{21.27}$$

substitute $\mathbf{K} = \mathbf{Y}\mathbf{Q}^{-1}$

$$\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^T + \mathbf{B}\mathbf{Y} + \mathbf{Y}^T\mathbf{B}^T < 0 \tag{21.28}$$

jointly find controller, stability

optimal: limit, optimize resources
adaptive: online modeling
robust: insensitive to model assumptions, system changes, faulty operation
nonlinear: geometrical, convex
Schur complement

$$\begin{pmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{pmatrix} > 0 \tag{21.29}$$

$\mathbf{R} > 0, \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T > 0$
Refs
Brogan: Modern Control Theory
Bertsekas: Nonlinear Programming
Zhou, Doyle, Glover: Robust and Optimal Control
Boyd, El Ghaoui, Feron, Balakrishnan: Linear Matrix Inequalities in System and Control Theory
Boyd and Vandenberghe: Convex Optimization
LQR
binary
min $|y(t) - y_r(t)|^2$
quadratic

$$\begin{bmatrix} \vec{u} \\ 1 \end{bmatrix}^T \begin{bmatrix} \mathbf{M} & \vec{v} \\ \vec{v}^T & s \end{bmatrix} \begin{bmatrix} \vec{u} \\ 1 \end{bmatrix} \tag{21.30}$$

$u_i \in \{-1, +1\}$
primal
combinatorial optimization
min $\vec{x}^T \mathbf{M} \vec{x}, x_i^2 - 1 = 0$
Lagrangian

$$\begin{aligned} \mathcal{L}(\vec{x}, \lambda) &= \vec{x}^T \mathbf{M} \vec{x} - \sum_i \lambda_i (x_i^2 - 1) \\ &= \vec{x}^T (\mathbf{M} - \mathbf{\Lambda}) \vec{x} + \mathrm{Tr}\Lambda \end{aligned} \tag{21.31}$$

dual
max $\mathrm{Tr}\Lambda, \mathbf{M} - \mathbf{\Lambda} \succeq 0$
continuous
Riccati
small gain theorem Kalman-Yakubovich-Popov Lemma H2 Hinf mu structured singular value
nonlinear
SDP Schur complement global linearization, time-varying linear Lie bracket quantum compute, control
distributed
subgradients, dual

LMI, feasibility
min norm control:

$$\vec{U}_n = -\mathbf{P}_n^T \left[\mathbf{P}_n \mathbf{P}_n^T\right]^{-1} \mathbf{A}^n \vec{x}_0 \tag{21.32}$$

$$\left[\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad \cdots \quad \mathbf{A}^{N-2^T}\mathbf{C}^T \quad \mathbf{A}^{N-1^T}\mathbf{C}^T\right] \tag{21.33}$$

State variable feedback
discrete time

$$\vec{u}_n = \mathbf{F}\vec{v}_n - \mathbf{K}\vec{x}_n \tag{21.34}$$

$$\vec{x}_{n+1} = [\mathbf{A} - \mathbf{B}\mathbf{K}]\vec{x}_n + \mathbf{B}\mathbf{F}\vec{v}_n \tag{21.35}$$

$$\Delta(\lambda) = |\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = 0 \tag{21.36}$$

$$[\mathbf{A} - \mathbf{B}\mathbf{K}]\vec{v}_i = \lambda_i \vec{v}_i \tag{21.37}$$

$$[\lambda_i\mathbf{I} - \mathbf{A} \quad \mathbf{B}] \left[\begin{array}{c} \vec{v}_i \\ \mathbf{K}\vec{v}_i \end{array}\right] = 0 \tag{21.38}$$

SVD basis nullspace
SVD solve for **K**
continuous time

$$\begin{aligned} \dot{\vec{x}} &= \mathbf{A}\vec{x} + \mathbf{B}(\vec{u} - \mathbf{K}\vec{x}) \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\vec{x} + \mathbf{B}\vec{u} \end{aligned} \tag{21.39}$$

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| &= |(\lambda\mathbf{I} - \mathbf{A})[\mathbf{I} + (\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{K}]| \\ &= |\lambda\mathbf{I} - \mathbf{A}||\mathbf{I} + (\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{K}| \end{aligned} \tag{21.40}$$

controllability implies sufficient rank
SVD
Observers

$$\vec{x}'_{n+1} = \mathbf{A}'\vec{x}'_n + \mathbf{E}\vec{y}_n + \vec{d}_n \tag{21.41}$$

$$\vec{e}_n = \vec{x}_n - \vec{x}'_n \tag{21.42}$$

$$\vec{e}_{n+1} = \mathbf{A}\vec{x}_n + \mathbf{B}\vec{u}_n - \mathbf{A}'\vec{x}'_n - \mathbf{EC}\vec{x}_n - \mathbf{ED}\vec{u}_n - \vec{d}_n \tag{21.43}$$

$$\vec{d} = (\mathbf{B} - \mathbf{ED})\vec{u}_n \tag{21.44}$$

$$\mathbf{A}' = \mathbf{A} - \mathbf{EC} \tag{21.45}$$

$$\vec{e}_{n+1} = \mathbf{A}'\vec{e}_n \tag{21.46}$$

$$|\lambda\mathbf{I} - \mathbf{A} + \mathbf{EC}| = 0 \tag{21.47}$$

system identification impulse frequency correlation
state estimation observer
optimal
dynamic programming
vector field
tangent space
Lie bracket
algebra
linear
car
linear car
car bracket
feedback linearization eliminate higher-order terms
non-holonomic (velocity constraint) Nyquist stability Bode plot lag, lead compensation ref to adaptive steppers Desoer's famous 1969 paper on the stability of slowly-varying systems controllability wronskian

21.1 PROBLEMS

(21.1) inverted pendulum

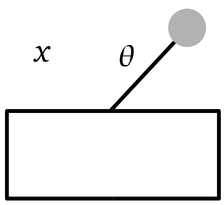

Figure 21.1. Cart

(21.2) PID problem

NMM Programming

Languages

symbolic mathematica maple interpreted APL Python scipy byte-code Java dyadic compiled Fortran C assembly FPGA GPU HDL VHDL Verilog ASIC DEShaw ALA

Libraries

C libs direct indirect NR Netlib linpack blas numpy

Graphics

high-low

python ipython matplot tk wx VTK scipy

java

gl

x

postscript

*ification

sdl stl