

5 Partial Differential Equations

Partial differential equations (PDEs) are equations that involve rates of change with respect to continuous variables. The configuration of a rigid body is specified by six numbers, but the configuration of a fluid is given by the continuous distribution of the temperature, pressure, and so forth. The dynamics for the rigid body take place in a finite-dimensional configuration space; the dynamics for the fluid occur in an infinite-dimensional configuration space. This distinction usually makes PDEs much harder to solve than ODEs, but here again there will be simple solutions for linear problems. Classic domains where PDEs are used include acoustics, fluid flow, electrodynamics, and heat transfer.

5.1 THE ORIGIN OF PARTIAL DIFFERENTIAL EQUATIONS

In the preceding chapter we saw how the solution for two coupled harmonic oscillators simplifies into two independent normal modes. What does the solution look like if there are 10 oscillators? 10^{10} ? Are there any simplifications? Not surprisingly, the answer is yes.

Consider an infinite chain of oscillators (Figure 5.1). The governing equation for the n th mass is

$$\begin{aligned}
 m\ddot{y}_n &= -k(y_n - y_{n+1}) - k(y_n - y_{n-1}) \\
 \ddot{y}_n &= \frac{k}{m}(y_{n+1} - 2y_n + y_{n-1}) \\
 &= \underbrace{k \delta x}_{\tau} \underbrace{\frac{\delta x}{m}}_{1/\rho} \frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x^2} .
 \end{aligned} \tag{5.1}$$

The two prefactors are the average spring constant τ and mass density ρ (remember that springs add inversely proportionally), and the final term is just an approximation to the second spatial derivative:

$$\begin{aligned}
 \frac{\partial y}{\partial x} &\approx \frac{y_{n-1} - y_n}{\delta x} \\
 \frac{\partial^2 y}{\partial x^2} &\approx \frac{1}{\delta x} \left[\frac{y_{n+1} - y_n}{\delta x} - \frac{y_n - y_{n-1}}{\delta x} \right] \\
 &= \frac{y_{n+1} - 2y_n + y_{n-1}}{\delta x^2} .
 \end{aligned} \tag{5.2}$$

Therefore in the limit of a small spacing between the springs, the system of ordinary differential equations for a chain of harmonic oscillators reduces to a single partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 y}{\partial x^2} \quad (5.3)$$

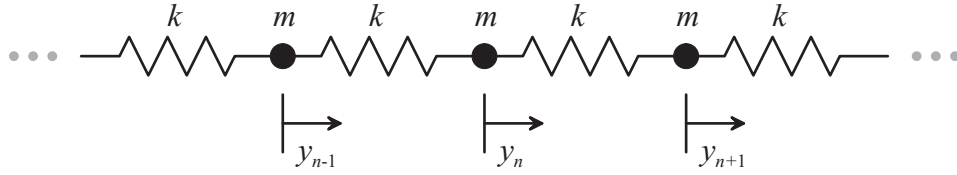


Figure 5.1. A chain of harmonic oscillators.

This equation is solved by a travelling wave. To see this, substitute a general solution $y = f(x + ct)$:

$$\begin{aligned} c^2 f'' &= \frac{\tau}{\rho} f'' \\ c &= \pm \sqrt{\frac{\tau}{\rho}} \end{aligned} \quad (5.4)$$

This represents an arbitrary disturbance travelling to the right and left with a velocity c : the location of the origin (for example) of f is determined by $x + ct = 0 \Rightarrow x/t = -c$. If there are nonlinearities the velocity will no longer be independent of the shape of the pulse: different wavelengths will travel at different speeds, a phenomenon called *dispersion*. Note that unlike the case for ODEs, the general solution involves an undetermined function and not just undetermined constants.

This same equation can be found directly by considering the transverse motion of an infinitesimal element of a continuous string that has a density of ρ and a tension τ (Figure 5.2). The governing equation for the transverse displacement y of this element is

$$\begin{aligned} ma &= F \\ \rho dx \frac{\partial^2 y}{\partial t^2} &= \tau \sin \theta \Big|_{x+dx} - \tau \sin \theta \Big|_x \\ &\approx \tau \tan \theta \Big|_{x+dx} - \tau \tan \theta \Big|_x \\ &\approx \tau \frac{\partial y}{\partial x} \Big|_{x+dx} - \tau \frac{\partial y}{\partial x} \Big|_x \\ \frac{\partial^2 y}{\partial t^2} &\approx \frac{\tau}{\rho} \frac{\partial^2 y}{\partial x^2} \end{aligned} \quad (5.5)$$

As a final example of the origin of partial differential equations, consider a highway for which position is measured by x , the density of cars by $\rho(x)$, and the rate at which cars pass a point by $I(x)$. In a time interval dt , the difference in the number of cars that enter and leave an interval dx is $[I(x) - I(x + dx)]dt$. This must be equal to the change in the number of cars at that interval $[\rho(t + dt) - \rho(t)]dx$ because the total number of

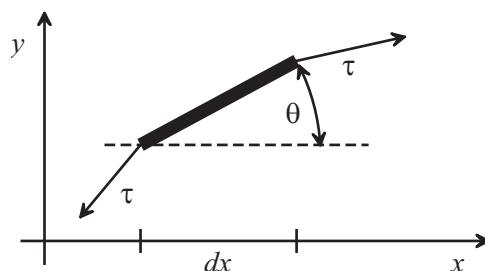


Figure 5.2. An infinitesimal piece of a string.

cars is (usually) conserved. Equating these gives

$$\begin{aligned} [\rho(t+dt) - \rho(t)]dx &= [I(x) - I(x+dx)]dt \\ \frac{\rho(t+dt) - \rho(t)}{dt} + \frac{I(x+dx) - I(x)}{dx} &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial I}{\partial x} &= 0 \quad . \end{aligned} \quad (5.6)$$

Now, assume that a driver's response can be modeled by a relationship between the traffic flow and density $I(x) = f(\rho(x), x)$. Multiplying both sides of equation (5.6) by $\partial I / \partial \rho$ gives

$$\begin{aligned} \frac{\partial \rho}{\partial t} \frac{\partial I}{\partial \rho} + \frac{\partial I}{\partial x} \frac{\partial I}{\partial \rho} &= 0 \\ \frac{\partial I}{\partial t} + \frac{\partial f(\rho, x)}{\partial \rho} \frac{\partial I}{\partial x} &= 0 \quad . \end{aligned} \quad (5.7)$$

Modeling traffic with a PDE can be a very good approximation, and can explain many observed traffic phenomena such as shock fronts and stationary disturbances [Whitham, 1974].

5.2 LINEAR PARTIAL DIFFERENTIAL EQUATIONS

As with ordinary differential equations, we will immediately specialize to linear partial differential equations, both because they occur so frequently and because they are amenable to analytical solution. A general linear second-order PDE for a field $\varphi(x, y)$ is

$$A \frac{\partial^2 \varphi}{\partial x^2} + B \frac{\partial^2 \varphi}{\partial x \partial y} + C \frac{\partial^2 \varphi}{\partial y^2} + D \frac{\partial \varphi}{\partial x} + E \frac{\partial \varphi}{\partial y} + F \varphi = G \quad , \quad (5.8)$$

where $G(x, y)$ is specified in some portion of the (x, y) plane and the solution must be determined in another portion.

A *characteristic* of a PDE is a surface across which there can be a discontinuity in the value or derivative of the solution. These define the domains which can be influenced by parts of the boundary conditions, much like the concept of a light cone in the space-time plane of special relativity [Taylor & Wheeler, 1992]. The characteristics of equation (5.8) are determined by the roots of a quadratic polynomial and accordingly can have the form of a hyperbola, a parabola, or an ellipse based on the sign of the discriminant $B^2 - 4AC$

[Hildebrand, 1976, Pearson, 1990]. The standard forms of these three cases define the most common PDEs that we will study:

- $B^2 - 4AC > 0$ (*hyperbolic*)

$$\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} \quad ; \quad (5.9)$$

- $B^2 - 4AC = 0$ (*parabolic*)

$$\nabla^2 \varphi = \frac{1}{D} \frac{\partial \varphi}{\partial t} \quad ; \quad (5.10)$$

- $B^2 - 4AC < 0$ (*elliptic*)

$$\nabla^2 \varphi = \rho \quad . \quad (5.11)$$

The first of these is a wave equation (like we found for the coupled harmonic oscillators), the second is a diffusion equation (for example, for heat or for ink), and the third is *Poisson's equation* (or *Laplace's equation* if the source term $\rho = 0$) and arises in boundary value problems (for example, for electric fields or for fluid flow).

5.3 SEPARATION OF VARIABLES

These three important partial differential equations can be reduced to systems of ordinary differential equations by the important technique of *separation of variables*. The logic of this technique may be confusing upon first acquaintance, but it rests on the uniqueness of solutions to differential equations: as with ODEs, if you can find *any* solution that solves the equation and satisfies the boundary conditions, then it is *the* solution. We will assume as an ansatz that the dependence of the solution on space and time can be written as a product of terms that each depend on a single coordinate, and then see if and how this can be made to solve the problem.

To start, the time dependence can be separated by assuming a solution of the form $\varphi(\vec{x}, t) = \psi(\vec{x})T(t)$. There is no time dependence for Laplace's equation; trying this in the diffusion equation gives

$$T(t)\nabla^2\psi(\vec{x}) = \frac{1}{D}\psi(\vec{x})\frac{\partial T(t)}{\partial t} \quad . \quad (5.12)$$

Dividing both sides by ψT results in no t dependence on the left hand side and no \vec{x} dependence on the right hand side, so both sides must be equal to some constant because the space and time variables can be varied arbitrarily. By convention, taking this constant to be $-k^2$ gives

$$\frac{1}{\psi(\vec{x})}\nabla^2\psi(\vec{x}) = \frac{1}{D}\frac{1}{T(t)}\frac{dT}{dt} = -k^2 \quad . \quad (5.13)$$

The t equation can immediately be integrated to find

$$T(t) = Ae^{-k^2Dt} \quad , \quad (5.14)$$

and the \vec{x} equation is *Helmholtz's equation*

$$\nabla^2\psi(\vec{x}) + k^2\psi(\vec{x}) = 0 \quad . \quad (5.15)$$

Similarly, for the wave equation this separation gives

$$\frac{1}{\psi(\vec{x})} \nabla^2 \psi(\vec{x}) = \frac{1}{c^2} \frac{1}{T} \frac{d^2 T}{dt^2} = -k^2 \quad . \quad (5.16)$$

The time equation is solved by

$$T(t) = A \sin(ckt) + B \cos(ckt) \quad , \quad (5.17)$$

and the space equation is Helmholtz's equation again.

Solving Helmholtz's equation will depend on the coordinate system used for the problem. There are three common ones used in 3D, based on the symmetry of the problem: rectangular, cylindrical, and spherical. Writing the derivative operators in each of these systems is a straightforward exercise in applying the chain rule to the coordinate definitions.

5.3.1 Rectangular Coordinates

Writing the *Laplacian* ∇^2 in rectangular coordinates leads to Helmholtz's equation as

$$\nabla^2 \psi + k^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \quad . \quad (5.18)$$

Assume that $\psi(\vec{x}) = X(x)Y(y)Z(z)$, substitute this in, and divide by it:

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} + k^2 = 0 \quad . \quad (5.19)$$

Since each term depends only on x , y , or z , the only way that this equation can hold is if each has a constant value (determined by the boundary conditions)

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = -k_1^2, \quad \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -k_2^2, \quad \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} = -k_3^2 \quad (5.20)$$

with $k_1^2 + k_2^2 + k_3^2 = k^2$. Each of these can be integrated to find

$$\begin{aligned} X &= A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \\ Y &= A_2 e^{ik_2 y} + B_2 e^{-ik_2 y} \\ Z &= A_3 e^{ik_3 z} + B_3 e^{-ik_3 z} \quad . \end{aligned} \quad (5.21)$$

Multiplying these back together, the spatial solution has the form

$$\psi(\vec{x}) = A e^{i\vec{k} \cdot \vec{x}} \quad (5.22)$$

with $\vec{k} \cdot \vec{k} = k^2$.

As an example, let's return to the 1D wave equation that we found from a chain of harmonic oscillators

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \quad . \quad (5.23)$$

With the separation $y(x, t) = X(x)T(t)$ this becomes

$$\frac{d^2 T}{dt^2} + c^2 k^2 T = 0 \quad \frac{d^2 X}{dx^2} + k^2 X = 0 \quad , \quad (5.24)$$

solved by

$$T = A \sin ckt + B \cos ckt \quad X = C \sin kx + D \cos kx \quad . \quad (5.25)$$

We know that the chain must be fixed at the ends ($X(0) = X(L) = 0$). This implies that $D = 0$, and that allowable values of the separation constant k are $k_n = n\pi/L$ for integer n . Therefore the general solution is

$$y(x, t) = \sum_n \sin\left(\frac{n\pi}{L}x\right) \left[A_n \sin\left(c\frac{n\pi}{L}t\right) + B_n \cos\left(c\frac{n\pi}{L}t\right) \right] \quad . \quad (5.26)$$

These are the normal modes of a string, with the oscillation frequency of each mode proportional to the number of cycles across the string.

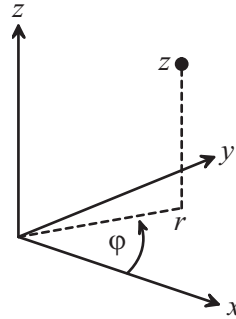


Figure 5.3. Cylindrical coordinate system.

5.3.2 Cylindrical Coordinates

In cylindrical coordinates (Figure 5.3), the Helmholtz equation is

$$\nabla^2 \psi + k^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \quad . \quad (5.27)$$

Once again, try separating by substituting in $\psi = R(r)\Phi(\varphi)Z(z)$ and dividing by it:

$$\frac{1}{R} \left[\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right] + \frac{1}{r^2} \frac{d^2 \Phi}{d\varphi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} + k^2 = 0 \quad . \quad (5.28)$$

The terms will cancel if

$$\begin{aligned} \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} &= -m^2 \\ \frac{1}{Z} \frac{d^2 Z}{dz^2} &= \alpha^2 - k^2 \\ \frac{1}{R} \left[\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right] - \frac{m^2}{r^2} + \alpha^2 &= 0 \end{aligned} \quad (5.29)$$

for constants α and m (these definitions are conventional). The first equation is easily solved:

$$\Phi = A \sin m\varphi + B \cos m\varphi \quad . \quad (5.30)$$

For the solution to be single valued $\Phi(\varphi + 2\pi) = \Phi(\varphi)$, and so m must be an integer. The second equation is similarly solved:

$$Z = C e^{z\sqrt{\alpha^2 - k^2}} + D e^{-z\sqrt{\alpha^2 - k^2}} \quad . \quad (5.31)$$

Rewriting the radial equation in terms of $r = \rho/\alpha$,

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(1 - \frac{m^2}{\rho^2} \right) R = 0 \quad . \quad (5.32)$$

This is *Bessel's equation*; its solution is given by Bessel functions

$$R = E J_m(\alpha r) + F N_m(\alpha r) \quad . \quad (5.33)$$

N_m is singular as $r \rightarrow 0$ while J_m is not, so if the solution is finite at the origin $F = 0$. If the radial solution must vanish for some r value it is necessary to know where the zeros of J_m occur; these are tabulated in many sources (such as [Abramowitz & Stegun, 1965]). The lowest ones are

$$\begin{aligned} J_0(x) = 0 &\Rightarrow x \approx 2.405, 5.520, 8.654, \dots \\ J_1(x) = 0 &\Rightarrow x \approx 3.832, 7.016, 10.173, \dots \\ J_2(x) = 0 &\Rightarrow x \approx 5.136, 8.417, 11.620, \dots \end{aligned} \quad . \quad (5.34)$$

If $\alpha = 0$, the radial equation becomes

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{m^2}{r^2} R = 0 \quad , \quad (5.35)$$

which is solved by

$$R(r) = \begin{cases} Gr^m + Hr^{-m} & (m \neq 0) \\ G + H \ln r & (m = 0) \end{cases} \quad . \quad (5.36)$$

5.3.3 Spherical Coordinates

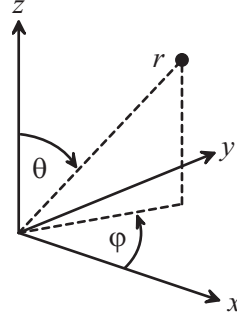


Figure 5.4. Spherical coordinate system.

Finally, in spherical coordinates (Figure 5.4) we want to separate

$$\nabla^2 \psi + k^2 \psi = 0$$

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) + \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right] + k^2 \psi = 0 . \quad (5.37)$$

Let's start with the radial part: $\psi = R(r)Y(\theta, \varphi)$

$$\frac{1}{R} \frac{1}{r} \frac{d^2}{dr^2} (rR) + \frac{1}{r^2 Y \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 Y}{\partial \varphi^2} \right] + k^2 = 0$$

$$\Rightarrow \frac{1}{Y \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 Y}{\partial \varphi^2} \right] = -\lambda ,$$

$$\frac{1}{R} \frac{1}{r} \frac{d^2}{dr^2} (rR) + k^2 - \frac{\lambda}{r^2} = 0 \quad (5.38)$$

for a constant λ . If $k^2 \neq 0$, substituting $r = \rho/k$ and then $R = S/\sqrt{\rho}$ gives Bessel's equation again:

$$\frac{d^2 S}{d\rho^2} + \frac{1}{\rho} \frac{dS}{d\rho} + \left(1 - \frac{\lambda + 1/4}{\rho^2} \right) S = 0 , \quad (5.39)$$

with the solution

$$R = A \frac{1}{\sqrt{kr}} J_{\sqrt{\lambda+1/4}}(kr) + B \frac{1}{\sqrt{kr}} N_{\sqrt{\lambda+1/4}}(kr) . \quad (5.40)$$

If $k^2 = 0$, the radial equation simplifies to

$$\frac{1}{r} \frac{d^2}{dr^2} (rR) - \frac{\lambda}{r^2} R = 0 , \quad (5.41)$$

solved by

$$R = A r^{(-1+\sqrt{1+4\lambda})/2} + B r^{(-1-\sqrt{1+4\lambda})/2} . \quad (5.42)$$

Now separate the angular parts with $Y = \Theta(\theta)\Phi(\varphi)$:

$$\begin{aligned} \frac{1}{\Theta} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\sin^2 \theta} \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} + \lambda &= 0 \\ \Rightarrow \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} &= -m^2 \quad , \\ \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left(\lambda - \frac{m^2}{\sin^2 \theta} \right) \Theta &= 0 \quad . \end{aligned} \quad (5.43)$$

The solution of the φ equation is

$$\Phi = Ae^{im\varphi} + Be^{-im\varphi} \quad . \quad (5.44)$$

For the θ equation, substitute $x = \cos \theta$:

$$\frac{d^2 \Theta}{dx^2} - \frac{2x}{1-x^2} \frac{d\Theta}{dx} + \frac{1}{1-x^2} \left[\lambda - \frac{m^2}{1-x^2} \right] \Theta = 0 \quad . \quad (5.45)$$

This is *Legendre's equation*, solved by the Legendre functions

$$\Theta = C\Theta_l^m(x) + DQ_l^m(x) \quad (5.46)$$

with $l(l+1) = \lambda$.

5.4 TRANSFORM TECHNIQUES

In the previous section we saw that separation of variables can turn partial differential equations into ordinary differential equations; this can also sometimes be done by taking the Fourier transform. For example, start with the PDE

$$\nabla^2 \varphi + A \frac{\partial^2 \varphi}{\partial t^2} + B \frac{\partial \varphi}{\partial t} = 0 \quad . \quad (5.47)$$

$\varphi(\vec{x}, t)$ is related to its Fourier transform $\Phi(\vec{k}, t)$ by

$$\begin{aligned} \varphi(\vec{x}, t) &= \int_{-\infty}^{\infty} \Phi(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} d\vec{k} \\ \Phi(\vec{k}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\vec{x}, t) e^{-i\vec{k} \cdot \vec{x}} d\vec{x} \quad . \end{aligned} \quad (5.48)$$

Substituting in the transform for φ in equation (5.47), exchanging the order of differentiation and integration, and grouping terms,

$$\int_{-\infty}^{\infty} \left[-k^2 \Phi(\vec{k}, t) + A \frac{\partial^2}{\partial t^2} \Phi(\vec{k}, t) + B \frac{\partial}{\partial t} \Phi(\vec{k}, t) \right] e^{i\vec{k} \cdot \vec{x}} d\vec{k} = 0 \quad . \quad (5.49)$$

The only way that the integral can equal zero for all x is if the integrand vanishes. The integrand now depends only on Φ and its time derivatives; if a solution can be found to this ordinary differential equation in t

$$-k^2 \Phi(\vec{k}, t) + A \frac{\partial^2}{\partial t^2} \Phi(\vec{k}, t) + B \frac{\partial}{\partial t} \Phi(\vec{k}, t) = 0 \quad (5.50)$$

then it will solve the integral equation. A solution to this differential equation can be multiplied by an arbitrary function of \vec{k} and still be a solution; this function is determined from initial conditions by the transform

$$\Phi(\vec{k}, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\vec{x}, 0) e^{-i\vec{k}\cdot\vec{x}} d\vec{x} \quad . \quad (5.51)$$

5.5 SELECTED REFERENCES

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Everything you ever wanted to know about wave equations.

5.6 PROBLEMS

- (3.1) Consider a round drumhead of radius L . For small displacements its motion is described by a linear wave equation. Find the frequencies of the six lowest oscillation modes, and plot the shape of the modes.
- (3.2) Solve a 1D diffusion equation with Fourier transforms.
- (3.3) Assume a crowded room full of generous children who have varying amounts of candy. Let $\varphi_{n,m}(t_i)$ be the amount of candy held by the n, m th child at time t_i . Because of the crowding, the children are approximately close-packed on a square grid. The children want to equalize the amount of candy, but it is so noisy that they can only talk to their nearest neighbors (although they are wearing watches). Find a simple strategy for them to use that results in the candy being evenly distributed, and in the continuum limit find a familiar PDE that is equivalent to this strategy.